Braids, hyperplane arrangements, and Milnor fibrations

Alex Suciu
Northeastern University

Workshop on Braids, Resolvent Degree and Hilbert’s 13th Problem
Institute for Pure and Applied Mathematics, UCLA
February 21, 2019
Let $X$ be a path-connected space. A *simple Weierstrass polynomial* of degree $n$ on $X$ is a map $f : X \times \mathbb{C} \to \mathbb{C}$ given by

$$f(x, z) = z^n + \sum_{i=1}^{n} a_i(x) z^{n-i},$$

with continuous coefficient maps $a_i : X \to \mathbb{C}$, and with no multiple roots for any $x \in X$.

Let $E = E(f) = \{(x, z) \in X \times \mathbb{C} \mid f(x, z) = 0\}$.

The restriction of $pr_1 : X \times \mathbb{C} \to X$ to $E$ defines an $n$-fold cover $\pi = \pi_f : E \to X$, the *polynomial covering map* associated to $f$. 

$$E \xleftarrow{\pi} X \times \mathbb{C} \xrightarrow{pr_1} X$$
**Configuration Spaces**

- Let $\text{Conf}_n(\mathbb{C}) = \{ z \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j \}$ and $U\text{Conf}_n(\mathbb{C}) = \text{Conf}_n(\mathbb{C}) / S_n$.

- Since $f : X \times \mathbb{C} \to \mathbb{C}$ has no multiple roots, the coefficient map $a = (a_1, \ldots, a_n) : X \to \mathbb{C}^n$ takes values in $\mathbb{C}^n \setminus \Delta_n = U\text{Conf}_n(\mathbb{C})$.

- Over $U\text{Conf}_n(\mathbb{C})$, there is a canonical $n$-fold polynomial covering map, $\pi_n : E(f_n) \to U\text{Conf}_n(\mathbb{C})$, determined by the W-polynomial $f_n(x, z) = z^n + \sum_{i=1}^{n} x_i z^{n-i}$.

- We get a pullback diagram of covers,

$$
\begin{array}{ccc}
E(f) & \longrightarrow & E(f_n) \\
\downarrow^{\pi_f} & & \downarrow^{\pi_n} \\
X & \longrightarrow & B^n \\
\end{array}
$$
Braid groups

Let $B_n$ be the Artin braid group on $n$ strands. Then $B_n = \pi_1(UConf_n(\mathbb{C}))$.

We let $\psi_n: B_n \hookrightarrow \text{Aut}(F_n)$ be the Artin representation.

The coefficient homomorphism, $\alpha = a_*: \pi_1(X) \to B_n$, is well-defined up to conjugacy.

Polynomial covers are those covers $\pi: E \to X$ for which the characteristic homomorphism $\chi: \pi_1(X) \to S_n$ factors through the canonical surjection $\tau_n: B_n \to S_n$.

$\begin{tikzcd}
\pi_1(X) \arrow[r, \chi] \arrow[dr, \alpha] & S_n \\
B_n \arrow[ur, \tau_n] &
\end{tikzcd}$
The root map

- Now assume that the W-polynomial $f$ completely factors as
  \[ f(x, z) = \prod_{i=1}^{n} (z - b_i(x)), \]
  with continuous roots $b_i : X \to \mathbb{C}$.

- Since $f$ is simple, the root map $b = (b_1, \ldots, b_n) : X \to \mathbb{C}^n$ takes values in $\text{Conf}_n(\mathbb{C})$.

- Over $\text{Conf}_n(\mathbb{C})$, there is a canonical $n$-fold cover, $\pi_{Qn} : E(Q_n) \to \text{Conf}_n(\mathbb{C})$, where
  \[ Q_n(w, z) = (z - w_1) \cdots (z - w_n). \]

- We get a pullback diagram of covers,

\[
\begin{array}{ccc}
E(f) & \xrightarrow{\pi_f} & E(Q_n) \\
\downarrow & & \downarrow \pi_{Qn} \\
X & \xrightarrow{b} & \text{Conf}_n(\mathbb{C})
\end{array}
\]
Braid bundles

- Let $P_n = \ker(\tau_n: B_n \to S_n)$ be the pure braid group. Then $P_n = \pi_1(\text{Conf}_n(\mathbb{C}))$.

- The map $\beta = b_*: \pi_1(X) \to P_n$ is well-defined up to conjugacy.

- The polynomial covers which are trivial covers are precisely those for which $\alpha = \iota_n \circ \beta$, where $\iota_n: P_n \hookrightarrow B_n$ is the inclusion map.

Theorem (D. Cohen, A.S. 1997)

Let $f: X \times \mathbb{C} \to \mathbb{C}$ be a simple W-polynomial. Let $Y = X \times \mathbb{C}\setminus E(f)$ and let $p: Y \to X$ be the restriction of $\text{pr}_1: X \times \mathbb{C} \to X$ to $Y$.

- The map $p: Y \to X$ is a locally trivial bundle, with structure group $B_n$ and fiber $\mathbb{C}_n = \mathbb{C}\setminus \{n \text{ points}\}$. Upon identifying $\pi_1(\mathbb{C}_n)$ with $F_n$, the monodromy of this bundle is $\psi_n \circ \alpha: \pi_1(X) \to \text{Aut}(F_n)$.

- If $f$ completely factors into linear factors, the structure group reduces to $P_n$, and the monodromy factors as $\psi_n \circ \iota_n \circ \beta$. 
Let $C$ be a reduced algebraic curve in $\mathbb{C}^2$, defined by a polynomial $f = f(z_1, z_2)$ of degree $n$.

Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a linear projection, and let $\mathcal{Y} = \{y_1, \ldots, y_s\}$ be the set of points in $\mathbb{C}$ for which the fibers of $\pi$ contain singular points of $C$, or are tangent to $C$.

WLOG, we may assume that $\pi = \text{pr}_1$ is generic with respect to $C$. That is, for each $k$, the line $\mathcal{L}_k = \pi^{-1}(y_k)$ contains at most one singular point $v_k$ of $C$ and does not belong to the tangent cone of $C$ at $v_k$, and, moreover, all tangencies are simple.

Let $\mathcal{L} = \bigcup \mathcal{L}_k$. 
POLYNOMIAL COVERS AND BRAID MONODROMY

BRAID MONODROMY OF PLANE ALGEBRAIC CURVES

$\mathbb{C}^2 \rightarrow \mathbb{C}$

$\pi$

ALEX SUCIU (NORTHEASTERN)

BRAIDS AND HYPERPLANE ARRANGEMENTS

IPAM, FEBRUARY 21, 2019 8 / 25
In the chosen coordinates, the defining polynomial $f$ of $C$ may be written as $f(x, z) = z^n + \sum_{i=1}^{n} a_i(x) z^{n-i}$.

Since $C$ is reduced, for each $x \notin \mathcal{Y}$, the equation $f(x, z) = 0$ has $n$ distinct roots. Thus, $f$ is a simple $W$-polynomial over $C \backslash \mathcal{Y}$, and

$$\pi = \pi_f : C \backslash C \cap \mathcal{L} \to C \backslash \mathcal{Y}$$

is the associated polynomial $n$-fold cover.

Note that $\mathcal{Y}(f) = ((C \backslash \mathcal{Y}) \times C) \backslash (C \backslash C \cap \mathcal{L}) = C^2 \backslash C \cup \mathcal{L}$.

Thus, the restriction of $\text{pr}_1$ to $\mathcal{Y}(f)$,

$$p : C^2 \backslash C \cup \mathcal{L} \to C \backslash \mathcal{Y},$$

is a bundle map, with structure group $B_n$, fiber $C_n$, and monodromy homomorphism $\alpha = a_* : \pi_1(C \backslash \mathcal{Y}) \to B_n$. 
Braid monodromy presentation

- The homotopy exact sequence of fibration \( p: \mathbb{C}^2 \setminus \mathcal{C} \cup \mathcal{L} \to \mathbb{C} \setminus \mathcal{Y} \):

\[
1 \longrightarrow \pi_1(\mathbb{C}n) \longrightarrow \pi_1(\mathbb{C}^2 \setminus \mathcal{C} \cup \mathcal{L}) \overset{p^*}{\longrightarrow} \pi_1(\mathbb{C} \setminus \mathcal{Y}) \longrightarrow 1.
\]

- This sequence is split exact, with action given by the braid monodromy homomorphism \( \alpha: \pi_1(\mathbb{C} \setminus \mathcal{Y}) \to \text{Aut}(\pi_1(\mathbb{C}n)) \).

- Order the points of \( \mathcal{Y} \) by decreasing real part, and pick the basepoint \( y_0 \) in \( \mathbb{C} \setminus \mathcal{Y} \) with \( \text{Re}(y_0) > \max\{\text{Re}(y_k)\} \).

- Choose loops \( \xi_k: [0, 1] \to \mathbb{C} \setminus \mathcal{Y} \) based at \( y_0 \), and going around \( y_k \).

- Setting \( x_k = [\xi_k] \), identify \( \pi_1(\mathbb{C} \setminus \mathcal{Y}, y_0) \) with \( F_s = \langle x_1, \ldots, x_s \rangle \).
  Similarly, identify \( \pi_1(\mathbb{C}n, \hat{y}_0) \) with \( F_n = \langle t_1, \ldots, t_n \rangle \).

- Then \( \pi_1(\mathbb{C}^2 \setminus \mathcal{C} \cup \mathcal{L}, \hat{y}_0) = F_n \rtimes_\alpha F_s \).
The corresponding presentation is

\[ \pi_1(\mathbb{C}^2 \setminus C \cup \mathcal{L}) = \langle t_1, \ldots, t_n, x_1, \ldots, x_s \mid x_k^{-1} t_i x_k = \alpha(x_k)(t_i) \rangle. \]

The group \( \pi_1(\mathbb{C}^2 \setminus C) \) is the quotient of \( \pi_1(\mathbb{C}^2 \setminus C \cup \mathcal{L}) \) by the normal closure of \( F_s = \langle x_1, \ldots, x_s \rangle \). Thus,

\[ \pi_1(\mathbb{C}^2 \setminus C) = \langle t_1, \ldots, t_n \mid t_i = \alpha(x_k)(t_i) \rangle. \]

This presentation can be simplified by Tietze-II moves to eliminate redundant relations. This yields the \textit{braid monodromy presentation}

\[ \pi_1(\mathbb{C}^2 \setminus C) = \langle t_1, \ldots, t_n \mid t_i = \alpha(x_k)(t_i), i = j_1, \ldots, j_{m_k - 1}; k = 1, \ldots, s \rangle. \]

where \( m_k \) is the multiplicity of the singular point \( y_k \).

(Libgober 1986) The 2-complex modeled on this presentation is homotopy equivalent to \( \mathbb{C}^2 \setminus C \).
Hyperplane arrangements

- An arrangement of hyperplanes is a finite collection $\mathcal{A}$ of codimension 1 linear (or affine) subspaces in $\mathbb{C}^\ell$.

- Intersection lattice $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.

- Complement: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$. It is a smooth, quasi-projective variety and also a Stein manifold. It has the homotopy type of a finite, connected, $\ell$-dimensional CW-complex.
FUNDAMENTAL GROUP

Example (The Boolean arrangement)

- \( B_n \): all coordinate hyperplanes \( z_i = 0 \) in \( \mathbb{C}^n \).
- \( L(B_n) \): Boolean lattice of subsets of \( \{0, 1\}^n \).
- \( M(B_n) \): complex algebraic torus \( (\mathbb{C}^*)^n \cong K(\mathbb{Z}^n, 1) \).

Example (The braid arrangement)

- \( A_n \): all diagonal hyperplanes \( z_i - z_j = 0 \) in \( \mathbb{C}^n \).
- \( L(A_n) \): lattice of partitions of \( [n] := \{1, \ldots, n\} \), ordered by refinement.
- \( M(A_n) = \text{Conf}_n(\mathbb{C}) \cong K(P_n, 1) \).

For an arbitrary (central) arrangement \( A \), let \( A' = \{ H \cap \mathbb{C}^2 \}_{H \in A} \) be a generic planar slice. Then the arrangement group, \( \pi = \pi_1(M(A)) \), is isomorphic to \( \pi_1(M(A')) \).
• So let $\mathcal{A}$ be an arrangement of $n$ affine lines in $\mathbb{C}^2$.

• Taking a generic projection $\mathbb{C}^2 \to \mathbb{C}$ yields the braid monodromy $\alpha = (\alpha_1, \ldots, \alpha_s)$, where $s = \#\{\text{multiple points}\}$; the braids $\alpha_r \in P_n$ can be read off the associated braided wiring diagram,

  ![Diagram of braided wiring diagram]

• The group $\pi = \pi_1(M(\mathcal{A}))$ has a presentation with meridional generators $x_1, \ldots, x_n$ and commutator relators $x_i \alpha_j(x_i)^{-1}$.

• Let $\pi/\gamma_k(\pi)$ be the $(k - 1)^{th}$ nilpotent quotient of $\pi$. Then:
  • $\pi_{ab} = \pi/\gamma_2$ equals $\mathbb{Z}^n$.
  • $\pi/\gamma_3$ is determined by $L(\mathcal{A})$.
  • $\pi/\gamma_4$ (and thus, $\pi$) is not determined by $L(\mathcal{A})$. (Rybnikov).
Cohomology ring

- The Betti numbers of the complement are given by
  \[ \sum_{q=0}^{\ell} b_q(M(\mathcal{A})) t^q = \sum_{X \in L(\mathcal{A})} \mu(X) (-t)^{\text{rank}(X)}, \]
  with \( \mu : L(\mathcal{A}) \to \mathbb{Z} \) given by \( \mu(\mathcal{C}^{\ell}) = 1 \) and \( \mu(X) = -\sum_{Y \supseteq X} \mu(Y) \).

- Let \( E = \bigwedge(\mathcal{A}) \) be the \( \mathbb{Z} \)-exterior algebra on degree-1 classes \( e_H \) dual to the meridians around the hyperplanes \( H \in \mathcal{A} \).

- Let \( \partial : E^* \to E^{*-1} \) be the differential given by \( \partial(e_H) = 1 \), and set \( e_B = \prod_{H \in B} e_H \) for each \( B \subset \mathcal{A} \).

- Building on work of Arnold & Brieskorn, Orlik and Solomon described the cohomology ring of \( M(\mathcal{A}) \) solely in terms of \( L(\mathcal{A}) \):
  \[ H^*(M(\mathcal{A}), \mathbb{Z}) \cong E/\langle \partial e_B \mid \text{codim} \bigcap_{H \in B} H < |B| \rangle. \]

- The space \( M(\mathcal{A}) \) is \( \mathbb{Q} \)-formal but not \( \mathbb{F}_p \)-formal in general.
Resonance Varieties

- Let $X$ be a connected, finite cell complex,
- Let $A = H^*(X, \mathbb{k})$, where $\text{char } \mathbb{k} \neq 2$. Then: $a \in A^1 \Rightarrow a^2 = 0$.
- We thus get a cochain complex

$$ (A, \cdot a): A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{} \cdots. $$

- The *resonance varieties* of $X$ are the jump loci for the cohomology of this complex

$$ \mathcal{R}_s^q(X, \mathbb{k}) = \{ a \in A^1 | \dim_{\mathbb{k}} H^q(A, \cdot a) \geq s \} $$

- E.g., $\mathcal{R}_1^1(X, \mathbb{k}) = \{ a \in A^1 | \exists b \in A^1, b \neq \lambda a, ab = 0 \}$.
- These loci are homogeneous subvarieties of $A^1 = H^1(X, \mathbb{k})$. In general, they can be arbitrarily complicated.
Resonance varieties of arrangements

Work of Arapura, Falk, D.Cohen, A.S., Libgober, and Yuzvinsky, completely describes the varieties $R_s(A) = R_s^1(M(A), \mathbb{C})$.

- $R_1(A)$ is a union of linear subspaces in $H^1(M(A), \mathbb{C}) \cong \mathbb{C}^{|A|}$.

- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.

- $R_s(A)$ is the union of those linear subspaces that have dimension at least $s + 1$.

- Each $k$-multinet on a sub-arrangement $B \subseteq A$ gives rise to a component of $R_1(A)$ of dimension $k - 1$. Moreover, all components of $R_1(A)$ arise in this way.
\( \mathcal{R}_1(\mathcal{A}) \subset \mathbb{C}^6 \) has 4 local components (from the triple points), and one essential component, from the above \((3, 2)\)-net:

\[
L_{124} = \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\}, \\
L_{135} = \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\}, \\
L_{236} = \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\}, \\
L_{456} = \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\}, \\
L = \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}.
\]
Let $X$ be a connected, finite cell complex, let $\pi = \pi_1(X, x_0)$, and let $\text{Hom}(\pi, \mathbb{C}^*)$ be the character variety of $X$ (the affine algebraic group of $\mathbb{C}$-valued, multiplicative characters on $\pi$).

The characteristic varieties of $X$ are the jump loci for homology with coefficients in rank-1 local systems on $X$:

$$\mathcal{V}_s^q(X) = \{ \rho \in \text{Hom}(\pi, \mathbb{C}^*) \mid \dim H_q(X, \mathbb{C}_\rho) \geq s \}.$$

These loci are Zariski closed subsets of the character variety. In general, they can be arbitrarily complicated.

The sets $\mathcal{V}_s^1(X)$ depend only on $\pi/\pi''$. 

ALEX SUCIU (NORTHEASTERN)  BRAIDS AND HYPERPLANE ARRANGEMENTS  IPAM, FEBRUARY 21, 2019
Let $\mathcal{A}$ be an arrangement of $n$ hyperplanes, and let $\text{Hom}(\pi_1(\mathcal{M}(\mathcal{A})), \mathbb{C}^*) = (\mathbb{C}^*)^n$ be the character torus.

The characteristic variety $\mathcal{V}_1(\mathcal{A}) := \mathcal{V}_1(\mathcal{M}(\mathcal{A}))$ lies in the subtorus $\{t \in (\mathbb{C}^*)^n \mid t_1 \cdots t_n = 1\}$; it is a finite union of torsion-translates of algebraic subtori of $(\mathbb{C}^*)^n$.

If a linear subspace $L \subset \mathbb{C}^n$ is a component of $\mathcal{R}_1(\mathcal{A})$, then the algebraic torus $T = \exp(L)$ is a component of $\mathcal{V}_1(\mathcal{A})$.

All components of $\mathcal{V}_1(\mathcal{A})$ passing through the origin $1 \in (\mathbb{C}^*)^n$ arise in this way (and thus, are combinatorially determined).

In general, though, there are translated subtori in $\mathcal{V}_1(\mathcal{A})$. 

Let $\mathcal{A}$ be a central arrangement in $\mathbb{C}^\ell$. For each $H \in \mathcal{A}$ let $\alpha_H$ be a linear form with $\ker(\alpha_H) = H$, and let $Q = \prod_{H \in \mathcal{A}} \alpha_H$.

$Q : \mathbb{C}^\ell \to \mathbb{C}$ restricts to a smooth fibration, $Q : M(\mathcal{A}) \to \mathbb{C}^*$. The Milnor fiber of the arrangement is $F(\mathcal{A}) := Q^{-1}(1)$.

$F$ is a Stein manifold. It has the homotopy type of a finite CW-complex of dimension $\ell - 1$.

In general, $F$ is not $\mathbb{Q}$-formal, and $H_*(F, \mathbb{Z})$ may have torsion.

$F$ is the regular, $\mathbb{Z}_n$-cover of $U = \mathbb{P}(M(\mathcal{A}))$, classified by the morphism $\pi_1(U) \to \mathbb{Z}_n$ taking each loop $x_H$ to 1 (where $n = |\mathcal{A}|$).
The monodromy diffeo, \( h: F \rightarrow F \), is given by \( h(z) = e^{2\pi i/n}z \).

Let \( \Delta(t) \) be the characteristic polynomial of \( h_*: H_1(F, \mathbb{C}) \). Since \( h^n = \text{id} \), we have

\[
\Delta(t) = \prod_{r|n} \Phi_r(t)^{e_r(\mathcal{A})},
\]

where \( \Phi_r(t) \) is the \( r \)-th cyclotomic polynomial, and \( e_r(\mathcal{A}) \in \mathbb{Z}_{\geq 0} \).

To compute \( h_* \), we may assume \( \ell = 3 \), so that \( \mathcal{A} = \mathbb{P}(\mathcal{A}) \) is an arrangement of lines in \( \mathbb{CP}^2 \).

If there is no point of \( \mathcal{A} \) of multiplicity \( q \geq 3 \) such that \( r \mid q \), then \( e_r(\mathcal{A}) = 0 \) (Libgober 2002).

In particular, if \( \mathcal{A} \) has only points of multiplicity 2 and 3, then
\[
\Delta(t) = (t - 1)^{n-1}(t^2 + t + 1)^{e_3}.
\]
If multiplicity 4 appears, then we also get factor of \( (t + 1)^{e_2} \cdot (t^2 + 1)^{e_4} \).
Let $A = H^\bullet (M(\mathcal{A}), \kappa)$, and let $\sigma = \sum_{H \in \mathcal{A}} e_H \in A^1$.

Assume $\kappa$ has characteristic $p > 0$, and define

$$\beta_p(\mathcal{A}) = \dim_{\kappa} H^1(\mathcal{A}, \cdot \sigma).$$

That is, $\beta_p(\mathcal{A}) = \max\{s \mid \sigma \in R^1_s(\mathcal{A}, \kappa)\}$.


$e_{pm}(\mathcal{A}) \leq \beta_p(\mathcal{A})$, for all $m \geq 1$.

**Theorem (Papadima–S. 2017)**

- Suppose $\mathcal{A}$ admits a $k$-net. Then $\beta_p(\mathcal{A}) = 0$ if $p \nmid k$ and $\beta_p(\mathcal{A}) \geq k - 2$, otherwise.
- If $\mathcal{A}$ admits a reduced $k$-multinet, then $e_k(\mathcal{A}) \geq k - 2$. 
**Theorem (PS)**

Suppose $\mathcal{A}$ has no points of multiplicity $3r$ with $r > 1$. Then $\mathcal{A}$ admits a reduced $3$-multinet iff $\mathcal{A}$ admits a $3$-net iff $\beta_3(\mathcal{A}) \neq 0$. Moreover,

- $\beta_3(\mathcal{A}) \leq 2$.
- $e_3(\mathcal{A}) = \beta_3(\mathcal{A})$, and thus $e_3(\mathcal{A})$ is combinatorially determined.

**Corollary**

Suppose all flats $X \in L_2(\mathcal{A})$ have multiplicity $2$ or $3$. Then $\Delta(t)$, and thus $b_1(F(\mathcal{A}))$, are combinatorially determined.

**Theorem (PS)**

Suppose $\mathcal{A}$ supports a $4$-net and $\beta_2(\mathcal{A}) \leq 2$. Then

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) = 2.$$
Conjecture (PS)

The characteristic polynomial of the degree 1 algebraic monodromy for the Milnor fibration of an arrangement $\mathcal{A}$ of rank at least 3 is given by the combinatorial formula

$$\Delta_\mathcal{A}(t) = (t - 1)|\mathcal{A}|^{-1}((t + 1)(t^2 + 1))^{\beta_2(\mathcal{A})}(t^2 + t + 1)^{\beta_3(\mathcal{A})}.$$ 

- The conjecture has been verified for
  - All sub-arrangements of non-exceptional Coxeter arrangements (Măcinic, Papadima).
  - All complex reflection arrangements (Măcinic, Papadima, Popescu, Dimca, Sticlaru).
  - Certain types of complexified real arrangements (Yoshinaga, Bailet, Torielli, Settepanella).
- A counterexample has been announced by Yoshinaga (2019):
  there is an arrangement of 16 planes in $\mathbb{C}^3$ with $e_2 = 0$ but $\beta_2 = 1$. 