1. **Fundamental groups in geometry**
   - Fundamental groups of manifolds
   - Kähler groups
   - Quasi-projective groups
   - Complements of hypersurfaces
   - Line arrangements
   - Artin groups

2. **Comparing classes of groups**
   - Kähler groups vs other groups
   - Quasi-projective groups vs other groups
Every finitely presented group $\pi$ can be realized as $\pi = \pi_1(M)$, for some smooth, compact, connected manifold $M^n$ of dim $n \geq 4$.

$M^n$ can be chosen to be orientable.

If $n$ even, $n \geq 4$, then $M^n$ can be chosen to be symplectic (Gompf).

If $n$ even, $n \geq 6$, then $M^n$ can be chosen to be complex (Taubes).

Requiring that $n = 3$ puts severe restrictions on the (closed) 3-manifold group $\pi = \pi_1(M^3)$. 

A Kähler manifold is a compact, connected, complex manifold, with a Hermitian metric $h$ such that $\omega = \text{im}(h)$ is a closed 2-form.

Smooth, complex projective varieties are Kähler manifolds.

A group $\pi$ is called a Kähler group if $\pi = \pi_1(M)$, for some Kähler manifold $M$.

The group $\pi$ is a projective group if $M$ can be chosen to be a projective manifold.

The classes of Kähler and projective groups are closed under finite direct products and passing to finite-index subgroups.

Every finite group is a projective group. [Serre ~1955]
The Kähler condition puts strong restrictions on $\pi$, e.g.:

- $\pi$ is finitely presented.
- $b_1(\pi)$ is even. [by Hodge theory]
- $\pi$ is 1-formal [Deligne–Griffiths–Morgan–Sullivan 1975]
- $\pi$ cannot split non-trivially as a free product. [Gromov 1989]

Problem: Are all Kähler groups projective groups?

Problem [Serre]: Characterize the class of projective groups.
A group $\pi$ is said to be a *quasi-Kähler group* if $\pi = \pi_1(M\backslash D)$, where $M$ is a Kähler manifold and $D$ is a divisor.

The group $\pi$ is a *quasi-projective group* if $M$ can be chosen to be a projective manifold.

$qK/qp$ groups are finitely presented. The classes of $qK/qp$ groups are closed under finite direct products and passing to finite-index subgroups.

For a $qp$ group $\pi$,
- $b_1(\pi)$ can be arbitrary (e.g., the free groups $F_n$).
- $\pi$ may be non-1-formal (e.g., the Heisenberg group).
- $\pi$ can split as a non-trivial free product (e.g., $F_2 = \mathbb{Z} \ast \mathbb{Z}$).

Problem: Are all quasi-Kähler groups quasi-projective groups?
Let $X$ be a quasi-Kähler manifold, and $G = \pi_1(X)$. Let $\{L_\alpha\}_{\alpha}$ be the non-zero irreducible components of $\mathcal{R}^1(G)$. If $G$ is 1-formal, then

- Each $L_\alpha$ is a linear subspace of $H^1(G, \mathbb{C})$.
- Each $L_\alpha$ is $p$-isotropic (i.e., restriction of $\cup G$ to $L_\alpha$ has rank $p$), with $\dim L_\alpha \geq 2p + 2$, for some $p = p(\alpha) \in \{0, 1\}$.
- If $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \{0\}$.
- $\mathcal{R}^1_k(G) = \{0\} \cup \bigcup_{\alpha : \dim L_\alpha > k + p(\alpha)} L_\alpha$.

Furthermore,

- If $X$ is compact, then $G$ is 1-formal, and each $L_\alpha$ is 1-isotropic.
- If $W_1(H^1(X, \mathbb{C})) = 0$, then $G$ is 1-formal, and each $L_\alpha$ is 0-isotropic.
A subclass of quasi-projective groups consists of fundamental groups of complements of hypersurfaces in $\mathbb{CP}^n$,

$$\pi = \pi_1(\mathbb{CP}^n \setminus \{ f = 0 \}), \quad f \in \mathbb{C}[z_0, \ldots, z_n] \text{ homogeneous.}$$

All such groups are 1-formal. [Kohno 1983]

By the Lefschetz hyperplane sections theorem, $\pi = \pi_1(\mathbb{CP}^2 \setminus C)$, for some plane algebraic curve $C$.

Zariski asked Van Kampen to find presentations for such groups.

Using the Alexander polynomial, Zariski showed that $\pi$ is not determined by the combinatorics of $C$ (number and type of singularities), but also depends on the position of its singularities.

**Problem (Zariski)**

Is $\pi = \pi_1(\mathbb{CP}^2 \setminus C)$ residually finite, i.e., is the map to the profinite completion, $\pi \rightarrow \pi^{\text{alg}} := \lim_{G \leftarrow_{\text{f.i.}} \pi} \pi / G$, injective?
Even more special are the *arrangement groups*, i.e., the fundamental groups of complements of complex hyperplane arrangements (or, equivalently, complex line arrangements).

Let $\mathcal{A}$ be an *arrangement of lines* in $\mathbb{CP}^2$, defined by a polynomial $f = \prod_{L \in \mathcal{A}} f_L$, with $f_L$ linear forms so that $L = \mathbb{P}(\ker(f_L))$.

The combinatorics of $\mathcal{A}$ is encoded in the *intersection poset*, $\mathcal{L}(\mathcal{A})$, with $\mathcal{L}_1(\mathcal{A}) = \{\text{lines}\}$ and $\mathcal{L}_2(\mathcal{A}) = \{\text{intersection points}\}$. 
Let $U(\mathcal{A}) = \mathbb{C}P^2 \setminus \bigcup_{L \in \mathcal{A}} L$. The group $\pi = \pi_1(U(\mathcal{A}))$ has a finite presentation with

- Meridional generators $x_1, \ldots, x_n$, where $n = |\mathcal{A}|$, and $\prod x_i = 1$.
- Commutator relators $x_i \alpha_j(x_i)^{-1}$, where $\alpha_1, \ldots, \alpha_s \in P_n \subset \text{Aut}(F_n)$, and $s = |\mathcal{L}_2(\mathcal{A})|$.

Let $\gamma_1(\pi) = \pi$, $\gamma_2(\pi) = \pi' = [\pi, \pi]$, $\gamma_k(\pi) = [\gamma_{k-1}(\pi), \pi]$, be the lower central series of $\pi$. Then:

- $\pi_{ab} = \pi/\gamma_2$ equals $\mathbb{Z}^{n-1}$.
- $\pi/\gamma_3$ is determined by $L(\mathcal{A})$.
- $\pi/\gamma_4$ (and thus, $\pi$) is not determined by $L(\mathcal{A})$ (G. Rybnikov).

**Problem (Orlik)**

Is $\pi$ torsion-free?

- Answer is yes if $U(\mathcal{A})$ is a $K(\pi, 1)$. This happens if the cone on $\mathcal{A}$ is a simplicial arrangement (Deligne), or supersolvable (Terao).
Let $\Gamma = (V, E)$ be a finite, simple graph, and let $\ell: E \to \mathbb{Z}_{\geq 2}$ be an edge-labeling. The associated Artin group:

$$A_{\Gamma, \ell} = \langle v \in V \mid \underbrace{v w v \cdots}_{\ell(e)} = \underbrace{w v w \cdots}_{\ell(e)}, \text{ for } e = \{v, w\} \in E \rangle.$$ 

If $(\Gamma, \ell)$ is Dynkin diagram of type $A_{n-1}$ with $\ell(\{i, i + 1\}) = 3$ and $\ell(\{i, j\}) = 2$ otherwise, then $A_{\Gamma, \ell}$ is the braid group $B_n$.

If $\ell(e) = 2$, for all $e \in E$, then

$$A_{\Gamma} = \langle v \in V \mid v w = w v \text{ if } \{v, w\} \in E \rangle.$$ 

is the right-angled Artin group associated to $\Gamma$.

$\Gamma \cong \Gamma' \iff A_{\Gamma} \cong A_{\Gamma'}$

[Kim–Makar-Limanov–Neggers–Roush 80 / Droms 87]
The corresponding Coxeter group,
\[ W_{\Gamma, \ell} = A_{\Gamma, \ell} / \langle v^2 = 1 \mid v \in V \rangle , \]
fits into exact sequence \[ 1 \rightarrow P_{\Gamma, \ell} \rightarrow A_{\Gamma, \ell} \rightarrow W_{\Gamma, \ell} \rightarrow 1 . \]

**THEOREM (Brieskorn 1971)**

*If* \( W_{\Gamma, \ell} \) *is finite, then* \( G_{\Gamma, \ell} \) *is quasi-projective.*

**Idea:** let
- \( A_{\Gamma, \ell} = \) reflection arrangement of type \( W_{\Gamma, \ell} \) (over \( \mathbb{C} \))
- \( X_{\Gamma, \ell} = \mathbb{C}^n \setminus \bigcup_{H \in A_{\Gamma, \ell}} H \), where \( n = |A_{\Gamma, \ell}| \)
- \( P_{\Gamma, \ell} = \pi_1(X_{\Gamma, \ell}) \)

then:
\[ A_{\Gamma, \ell} = \pi_1(X_{\Gamma, \ell}/W_{\Gamma, \ell}) = \pi_1(\mathbb{C}^n \setminus \{\delta_{\Gamma, \ell} = 0\}) \]

**THEOREM (Kapovich–Millson 1998)**

*There exist infinitely many* \((\Gamma, \ell)\) *such that* \( A_{\Gamma, \ell} \) *is not quasi-projective.*
Kähler groups vs other groups

**Question (Donaldson–Goldman 1989)**

Which 3-manifold groups are Kähler groups?

Reznikov gave a partial solution in 2002.

**Theorem (Dimca–S. 2009)**

Let $G$ be the fundamental group of a closed 3-manifold. Then $G$ is a Kähler group $\iff \pi$ is a finite subgroup of $O(4)$, acting freely on $S^3$.

- Idea of our proof: compare the resonance varieties of 3-manifolds to those of Kähler manifolds.

- By passing to a suitable index-2 subgroup of $G$, we may assume that the closed 3-manifold is orientable.
PROPOSITION

Let $M$ be a closed, orientable 3-manifold. Then:

1. $H^1(M, \mathbb{C})$ is not 1-isotropic.
2. If $b_1(M)$ is even, then $\mathcal{R}_1^1(M) = H^1(M, \mathbb{C})$.

On the other hand, it follows from a previous theorem that:

PROPOSITION

Let $M$ be a compact Kähler manifold with $b_1(M) \neq 0$. If $\mathcal{R}_1^1(M) = H^1(M, \mathbb{C})$, then $H^1(M, \mathbb{C})$ is 1-isotropic.

- If $G$ is a Kähler, then $b_1(G)$ even.
- Thus, if $G$ is both a 3-mfd group and a Kähler group $\implies b_1(G) = 0$.
- Using work of Fujiwara (1999) and Reznikov (2002) on Kazhdan’s property (T), as well as Perelman (2003), it follows that $G$ is a finite subgroup of $O(4)$.
Alternative proofs have later been given by Kotschick (2012) and Biswas, Mj, and Seshadri (2012).

**Theorem (Friedl–S. 2014)**

Let $N$ be a 3-manifold with non-empty, toroidal boundary. If $\pi_1(N)$ is a Kähler group, then $N \cong S^1 \times S^1 \times I$.

Subsequent generalization by Kotschick (dropping the toroidal boundary assumption): If $G$ is both an infinite 3-manifold group and a Kähler group, then $G$ is a surface group.
Theorem (DPS 2009)

Let $\Gamma$ be a finite simple graph, and let $A_\Gamma$ be the corresponding RAAG. The following are equivalent:

1. $A_\Gamma$ is a Kähler group.
2. $A_\Gamma$ is a free abelian group of even rank.
3. $\Gamma$ is a complete graph on an even number of vertices.

Theorem (S. 2011)

Let $A$ be an arrangement of lines in $\mathbb{CP}^2$, with group $\pi = \pi_1(U(A))$. The following are equivalent:

1. $\pi$ is a Kähler group.
2. $\pi$ is a free abelian group of even rank.
3. $A$ consists of an odd number of lines in general position.
Let $\pi$ be the fundamental group of a closed, orientable 3-manifold. Assume $\pi$ is 1-formal. Then the following are equivalent:

1. $m(\pi) \cong m(\pi_1(X))$, for some quasi-projective manifold $X$.
2. $m(\pi) \cong m(\pi_1(N))$, where $N$ is either $S^3$, $\#^n S^1 \times S^2$, or $S^1 \times \Sigma_g$.

Let $N$ be a 3-mfd with empty or toroidal boundary. If $\pi_1(N)$ is a quasi-projective group, then all prime components of $N$ are graph manifolds.

In particular, the fundamental group of a hyperbolic 3-manifold with empty or toroidal boundary is never a qp-group.
Comparing classes of groups

Comparing classes of groups

Quasi-projective groups vs other groups

Theorem (DPS 2009)

A right-angled Artin group $A_{\Gamma}$ is a quasi-projective group if and only if $\Gamma$ is a complete multipartite graph $K_{n_1, \ldots, n_r} = \overline{K}_{n_1} \ast \cdots \ast \overline{K}_{n_r}$, in which case $A_{\Gamma} = F_{n_1} \times \cdots \times F_{n_r}$.

Theorem (S. 2011)

Let $\pi = \pi_1(U(\mathcal{A}))$ be an arrangement group. The following are equivalent:

1. $\pi$ is a RAAG.
2. $\pi$ is a finite direct product of finitely generated free groups.
3. $G(\mathcal{A})$ is a forest.

Here $G(\mathcal{A})$ is the ‘multiplicity’ graph, with

- vertices: points $P \in L_2(\mathcal{A})$ with multiplicity at least 3;
- edges: $\{P, Q\}$ if $P, Q \in L$, for some $L \in \mathcal{A}$. 