1. Resonance varieties of CDGAs
   - Commutative differential graded algebras
   - Resonance varieties
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2. Resonance varieties of spaces
   - Algebraic models for spaces
   - Germs of jump loci
   - Tangent cones and exponential maps
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3. Infinitesimal finiteness obstructions
   - Spaces with finite models
   - Associated graded Lie algebras
   - Holonomy Lie algebras
   - Malcev Lie algebras
   - Finiteness obstructions for groups
Let $A = (A^\bullet, d)$ be a commutative, differential graded algebra over a field $\mathbb{k}$ of characteristic 0. That is:

- $A = \bigoplus_{i \geq 0} A^i$, where $A^i$ are $\mathbb{k}$-vector spaces.
- The multiplication $\cdot : A^i \otimes A^j \rightarrow A^{i+j}$ is graded-commutative, i.e., $ab = (-1)^{|a||b|} ba$ for all homogeneous $a$ and $b$.
- The differential $d : A^i \rightarrow A^{i+1}$ satisfies the graded Leibnitz rule, i.e., $d(ab) = d(a)b + (-1)^{|a|} a d(b)$.

A CDGA $A$ is of \textit{finite-type} (or \textit{q-finite}) if

- it is connected (i.e., $A^0 = \mathbb{k} \cdot 1$);
- $\dim_{\mathbb{k}} A^i$ is finite for $i \leq q$.

Let $H^i(A) = \ker(d: A^i \rightarrow A^{i+1})/\mathrm{im}(d: A^{i-1} \rightarrow A^i)$. Then $H^\bullet(A)$ inherits an algebra structure from $A$. 
A cdga morphism $\varphi: A \to B$ is both an algebra map and a cochain map. Hence, it induces a morphism $\varphi^*: H^\bullet(A) \to H^\bullet(B)$.

A map $\varphi: A \to B$ is a quasi-isomorphism if $\varphi^*$ is an isomorphism. Likewise, $\varphi$ is a $q$-quasi-isomorphism (for some $q \geq 1$) if $\varphi^*$ is an isomorphism in degrees $\leq q$ and is injective in degree $q + 1$.

Two cdgas, $A$ and $B$, are $(q)$-equivalent ($\sim_q$) if there is a zig-zag of $(q)$-quasi-isomorphisms connecting $A$ to $B$.

A cdga $A$ is formal (or just $q$-formal) if it is $(q)$-equivalent to $(H^\bullet(A), d = 0)$. 

Since $A$ is connected and $d(1) = 0$, we have $Z^1(A) = H^1(A)$.

For each $a \in Z^1(A)$, we construct a cochain complex,

$$(A^\bullet, \delta_a): \ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

with differentials $\delta_a^i(u) = a \cdot u + d(u)$, for all $u \in A^i$.

The resonance varieties of $A$ are the sets

$$\mathcal{R}_k^i(A) = \{ a \in H^1(A) \mid \dim H^i(A^\bullet, \delta_a) \geq k \}.$$

If $A$ is $q$-finite, then $\mathcal{R}_k^i(A)$ are algebraic varieties for all $i \leq q$.

If $A$ is a CDGA (so that $d = 0$), these varieties are homogeneous subvarieties of $H^1(A) = A^1$. 
Fix a $k$-basis $\{e_1, \ldots, e_r\}$ for $H^1(A)$, and let $\{x_1, \ldots, x_r\}$ be the dual basis for $H_1(A) = (H^1(A))^*$. 

Identify $\text{Sym}(H_1(A))$ with $S = k[x_1, \ldots, x_r]$, the coordinate ring of the affine space $H^1(A)$. 

Define a cochain complex of free $S$-modules, $L(A) := (A^* \otimes_k S, \delta)$,

$$
\cdots \longrightarrow A^i \otimes S \overset{\delta^i}{\longrightarrow} A^{i+1} \otimes S \overset{\delta^{i+1}}{\longrightarrow} A^{i+2} \otimes S \longrightarrow \cdots,
$$

where $\delta^i(u \otimes f) = \sum_{j=1}^n e_j u \otimes fx_j + d u \otimes f$. 

The specialization of $(A \otimes_k S, \delta)$ at $a \in A^1$ coincides with $(A, \delta_a)$. 

Hence, $R^i_k(A)$ is the zero-set of the ideal generated by all minors of size $b_i(A) - k + 1$ of the block-matrix $\delta^{i+1} \oplus \delta^i$. 

In particular, $R^1_k(A) = V(I_{r-k}(\delta^1))$, the zero-set of the ideal of codimension $k$ minors of $\delta^1$. 
**Example (Exterior Algebra)**

Let $E = \bigwedge V$, where $V = \mathbb{k}^n$, and $S = \text{Sym}(V)$. Then $L(E)$ is the Koszul complex on $V$. E.g., for $n = 3$: 

$$
\delta^1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \quad \delta^2 = \begin{pmatrix} x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \\ 0 & -x_1 & -x_2 \end{pmatrix} 
$$

$$
\begin{array}{c}
S \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
S^3 \rightarrow S
\end{array}
$$

Hence,

$$
\mathcal{R}^i_k(E) = \begin{cases} 
\{0\} & \text{if } k \leq \binom{n}{i}, \\
\emptyset & \text{otherwise}.
\end{cases}
$$
Example (Non-zero resonance)

Let $A = \wedge (e_1, e_2, e_3)/\langle e_1 e_2 \rangle$, and set $S = \mathbb{k}[x_1, x_2, x_3]$. Then

$$\delta^1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \delta^2 = \begin{pmatrix} x_3 & 0 & -x_1 \\ 0 & x_3 & -x_2 \end{pmatrix}$$

$L(A) : S \to S^3 \to S^2$.

$$\mathcal{R}_k^1(A) = \begin{cases} \{x_3 = 0\} & \text{if } k = 1, \\
\{0\} & \text{if } k = 2 \text{ or } 3, \\
\emptyset & \text{if } k > 3. \end{cases}$

Example (Non-linear resonance)

Let $A = \wedge (e_1, \ldots, e_4)/\langle e_1 e_3, e_2 e_4, e_1 e_2 + e_3 e_4 \rangle$. Then

$$\delta^1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \delta^2 = \begin{pmatrix} x_4 & 0 & 0 & -x_1 \\ 0 & x_3 & -x_2 & 0 \\ -x_2 & x_1 & x_4 & -x_3 \end{pmatrix}$$

$L(A) : S \to S^4 \to S^3$.

$$\mathcal{R}_1^1(A) = \{x_1 x_2 + x_3 x_4 = 0\}$
**Resonance varieties of CDGAs**

**Example (Non-homogeneous resonance)**

- Let $A = \wedge(a, b)$ with $d\ a = 0$, $d\ b = b \cdot a$.
- $H^1(A) = \mathbb{C}$, generated by $a$. Set $S = \mathbb{C}[x]$. Then:

$$L(A) : S \xrightarrow{\delta^1 = \begin{pmatrix} 0 \\ x \end{pmatrix}} S^2 \xrightarrow{\delta^2 = (x-1 \ 0)} S \ .$$

- Hence, $\mathcal{R}^1(A) = \{0, 1\}$, a non-homogeneous subvariety of $\mathbb{C}$.
- Let $A'$ be the sub-CDGA generated by $a$. The inclusion map, $A' \hookrightarrow A$, induces an isomorphism in cohomology.
- But $\mathcal{R}^1(A') = \{0\}$, and so the resonance varieties of $A$ and $A'$ differ, although $A$ and $A'$ are quasi-isomorphic.

**Proposition**

If $A \simeq_q A'$, then $\mathcal{R}^i_k(A)_{(0)} \cong \mathcal{R}^i_k(A')_{(0)}$, for all $i \leq q$ and $k \geq 0$. 

**Cohomology jump loci**

Alex Suciu (Northeastern)
**Theorem (Budur–Rubio, Denham–S. 2018)**

If $A$ is a connected $k$-CDGA $A$ with locally finite cohomology, then

$$\text{TC}_0(\mathcal{R}_k^i(A)) \subseteq \mathcal{R}_k^i(H^\bullet(A)).$$

In general, we cannot replace $\text{TC}_0(\mathcal{R}_k^i(A))$ by $\mathcal{R}_k^i(A)$.

**Example**

- Let $A = \wedge(a, b)$ with $\text{d}a = 0$ and $\text{d}b = b \cdot a$.
- Then $H^\bullet(A) = \wedge(a)$, and so $\mathcal{R}_1^1(A) = \{0\}$.
- Hence $\mathcal{R}_1^1(A) = \{0, 1\}$ is *not* contained in $\mathcal{R}_1^1(A)$, though $\text{TC}_0(\mathcal{R}_1^1(A)) = \{0\}$ is.
In general, the inclusion $\mathcal{TC}_0(\mathcal{R}_k^i(A)) \subseteq \mathcal{R}_k^i(H^\bullet(A))$ is strict.

**Example**

- Let $A = \bigwedge (a, b, c)$ with $d\ a = d\ b = 0$ and $d\ c = a \wedge b$.

- Writing $S = \mathbb{k}[x, y]$, we have:

  $\delta^1 = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$  \quad $\delta^2 = \begin{pmatrix} y - x & 1 \\ 0 & 0 & -x \\ 0 & 0 & -y \end{pmatrix}$

  $L(A) : S \xrightarrow{} S^3 \xrightarrow{} S^3$.

- Hence $\mathcal{R}_1^1(A) = \{0\}$.

- But $H^\bullet(A) = \bigwedge (a, b)/(ab)$, and so $\mathcal{R}_1^1(H^\bullet(A)) = \mathbb{k}^2$. 
Given any space $X$, there is an associated Sullivan $\mathbb{Q}$-cdga, $A_{PL}(X)$, such that $H^\bullet(A_{PL}(X)) = H^\bullet(X, \mathbb{Q})$.

We say $X$ is $q$-finite if $X$ has the homotopy type of a connected CW-complex with finite $q$-skeleton, for some $q \geq 1$.

An algebraic ($q$-)model (over $k$) for $X$ is a $k$-cgda $(A, d)$ which is ($q$-) equivalent to $A_{PL}(X) \otimes_{\mathbb{Q}} k$.

If $M$ is a smooth manifold, then $\Omega_{dR}(M)$ is a model for $M$ (over $\mathbb{R}$).

Examples of spaces having finite-type models include:

- Formal spaces (such as compact Kähler manifolds, hyperplane arrangement complements, toric spaces, etc).
- Smooth quasi-projective varieties, compact solvmanifolds, Sasakian manifolds, etc.
**Germs of Jump Loci**

**Theorem (Dimca–Papadima 2014)**

Let $X$ be a $q$-finite space, and suppose $X$ admits a $q$-finite, $q$-model $A$. Then the map $\exp : H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^*)$ induces a local analytic isomorphism $H^1(A)_{(0)} \to \text{Char}(X)_{(1)}$, which identifies the germ at 0 of $R^i_k(A)$ with the germ at 1 of $V^i_k(X)$, for all $i \leq q$ and $k \geq 0$.

**Corollary**

If $X$ is a $q$-formal space, then $V^i_k(X)_{(1)} \cong R^i_k(X)_{(0)}$, for $i \leq q$ and $k \geq 0$.

- A precursor to corollary can be found in work of Green, Lazarsfeld, and Ein on cohomology jump loci of compact Kähler manifolds.
- The case when $q = 1$ was first established in [DPS 2019].
Tangent cones and exponential maps

The map \( \exp : \mathbb{C}^n \to (\mathbb{C}^\times)^n, (z_1, \ldots, z_n) \mapsto (e^{z_1}, \ldots, e^{z_n}) \) is a homomorphism taking 0 to 1.

For a Zariski-closed subset \( W = V(I) \) inside \((\mathbb{C}^\times)^n\), define:
- The tangent cone at 1 to \( W \) as \( TC_1(W) = V(\mathrm{in}(I)) \).
- The exponential tangent cone at 1 to \( W \) as

\[
\tau_1(W) = \{ z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \forall \lambda \in \mathbb{C} \}
\]

These sets are homogeneous subvarieties of \( \mathbb{C}^n \), which depend only on the analytic germ of \( W \) at 1.

Both commute with finite unions and arbitrary intersections.

\( \tau_1(W) \subseteq TC_1(W) \).
- = if all irred components of \( W \) are subtori.
- \( \neq \) in general.

(DPS 2009) \( \tau_1(W) \) is a finite union of rationally defined subspaces.
The tangent cone theorem

Let $X$ be a connected CW-complex with finite $q$-skeleton.

**Theorem (Libgober 2002, DPS 2009)**

For all $i \leq q$ and $k \geq 0$,

$$
\tau_1(\mathcal{V}_k^i(X)) \subseteq TC_1(\mathcal{V}_k^i(X)) \subseteq R_k^i(X).
$$

**Theorem (DPS-2009, DP-2014)**

Suppose $X$ is a $q$-formal space. Then, for all $i \leq q$ and $k \geq 0$,

$$
\tau_1(\mathcal{V}_k^i(X)) = TC_1(\mathcal{V}_k^i(X)) = R_k^i(X).
$$

In particular, all irreducible components of $R_k^i(X)$ are rationally defined linear subspaces of $H^1(X, \mathbb{C})$. 
Detecting non-formality

Example

Let $\pi = \langle x_1, x_2 \mid [x_1, [x_1, x_2]] \rangle$. Then $V_1^1(\pi) = \{ t_1 = 1 \}$, and so

$$\tau_1(V_1^1(\pi)) = TC_1(V_1^1(\pi)) = \{ x_1 = 0 \}.$$ 

On the other hand, $R_1^1(\pi) = \mathbb{C}^2$, and so $\pi$ is not 1-formal.

Example

Let $\pi = \langle x_1, \ldots, x_4 \mid [x_1, x_2], [x_1, x_4][x_2^{-2}, x_3], [x_1^{-1}, x_3][x_2, x_4] \rangle$. Then

$$R_1^1(\pi) = \{ z \in \mathbb{C}^4 \mid z_1^2 - 2z_2^2 = 0 \}.$$ 

This is a quadric hypersurface which splits into two linear subspaces over $\mathbb{R}$, but is irreducible over $\mathbb{Q}$. Thus, $\pi$ is not 1-formal.
**Example**

Let $\pi$ be a finitely presented group with $\pi_{ab} = \mathbb{Z}^3$ and

$$\mathcal{V}_1^1(\pi) = \{(t_1, t_2, t_3) \in (\mathbb{C}^*)^3 \mid (t_2 - 1) = (t_1 + 1)(t_3 - 1)\},$$

This is a complex, 2-dimensional torus passing through the origin, but this torus does not embed as an algebraic subgroup in $(\mathbb{C}^*)^3$. Indeed,

$$\tau_1(\mathcal{V}_1^1(\pi)) = \{x_2 = x_3 = 0\} \cup \{x_1 - x_3 = x_2 - 2x_3 = 0\}.$$ 

Hence, $\pi$ is not 1-formal.
Example

- Let $\text{Conf}_n(E)$ be the configuration space of $n$ labeled points of an elliptic curve $E = \Sigma_1$.

- Using the computation of $H^\bullet(\text{Conf}_n(\Sigma_g), \mathbb{C})$ by Totaro (1996), we find that $\mathcal{R}_1(\text{Conf}_n(E))$ is equal to

$$\left\{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \mid \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 0, \quad x_i y_j - x_j y_i = 0, \text{ for } 1 \leq i < j \leq n \right\}$$

- For $n \geq 3$, this is an irreducible, non-linear variety (a rational normal scroll). Hence, $\text{Conf}_n(E)$ is not 1-formal.
**THEOREM (EXPONENTIAL AX–LINDEMANN THEOREM)**

Let $V \subseteq \mathbb{C}^n$ and $W \subseteq (\mathbb{C}^*)^n$ be irreducible algebraic subvarieties.

1. Suppose $\dim V = \dim W$ and $\exp(V) \subseteq W$. Then $V$ is a translate of a linear subspace, and $W$ is a translate of an algebraic subtorus.

2. Suppose the exponential map $\exp: \mathbb{C}^n \to (\mathbb{C}^*)^n$ induces a local analytic isomorphism $V_{(0)} \to W_{(1)}$. Then $W_{(1)}$ is the germ of an algebraic subtorus.

**THEOREM (BUDUR–WANG 2017)**

If $X$ is a $q$-finite space which admits a $q$-finite $q$-model, then, for all $i \leq q$ and $k \geq 0$, the irreducible components of $\mathcal{V}_k^i(X)$ passing through 1 are algebraic subtori of $\text{Char}(X)$. 
**Example**

Let $G$ be a f.p. group with $G_{ab} = \mathbb{Z}^n$ and $\mathcal{V}_1^1(G) = \{ t \in (\mathbb{C}^\times)^n \mid \sum_{i=1}^n t_i = n \}$. Then $G$ admits no 1-finite 1-model.

**Theorem (Papadima–S. 2017)**

Suppose $X$ is $(q + 1)$ finite, or $X$ admits a $q$-finite $q$-model. Let $\mathcal{M}_q(X)$ be Sullivan’s $q$-minimal model of $X$. Then $b_i(\mathcal{M}_q(X)) < \infty$, $\forall i \leq q + 1$.

**Corollary**

Let $G$ be a f.g. group. Assume that either $G$ is finitely presented, or $G$ has a 1-finite 1-model. Then $b_2(\mathcal{M}_1(G)) < \infty$.

**Example**

Let $G = F_n / F_n''$ with $n \geq 2$. We have $\mathcal{V}_1^1(G) = \mathcal{V}_1^1(F_n) = (\mathbb{C}^\times)^n$, and so $G$ passes the Budur–Wang test. But $b_2(\mathcal{M}_1(G)) = \infty$, and so $G$ admits no 1-finite 1-model (and is not finitely presented).
The *lower central series* of a group $G$ is defined inductively by

$$\gamma_1 G = G \quad \text{and} \quad \gamma_{k+1} G = [\gamma_k G, G].$$

This forms a filtration of $G$ by characteristic subgroups. The LCS quotients, $\gamma_k G/\gamma_{k+1} G$, are abelian groups.

The group commutator induces a graded Lie algebra structure on

$$\text{gr}(G, k) = \bigoplus_{k \geq 1} (\gamma_k G/\gamma_{k+1} G) \otimes \mathbb{Z} k.$$

Assume $G$ is finitely generated. Then $\text{gr}(G)$ is also finitely generated (in degree 1) by $\text{gr}_1(G) = H_1(G, k)$.

For instance, $\text{gr}(F_n)$ is the free graded Lie algebra $\mathbb{L}_n := \text{Lie}(k^n)$. 

**Alex Suciu (Northeastern)**

Cohomology jump loci

MIMS Summer School 2018
Holonomy Lie algebras

- Let $A$ be a 1-finite cdga. Set $A_i = (A^i)^* = \text{Hom}_k(A^i, k)$.

- Let $\mu^*: A_2 \to A_1 \wedge A_1$ be the dual to the multiplication map $\mu: A^1 \wedge A^1 \to A^2$.

- Let $d^*: A_2 \to A_1$ be the dual of the differential $d: A^1 \to A^2$.

- The holonomy Lie algebra of $A$ is the quotient
  \[ \mathfrak{h}(A) = \text{Lie}(A_1)/\langle \text{im}(\mu^* + d^*) \rangle. \]

- For a f.g. group $G$, set $\mathfrak{h}(G) := \mathfrak{h}(H^\bullet(G, k))$. There is then a canonical surjection $\mathfrak{h}(G) \to \text{gr}(G)$, which is an isomorphism precisely when $\text{gr}(G)$ is quadratic.
MALCEV LIE ALGEBRAS

The group-algebra $kG$ has a natural Hopf algebra structure, with comultiplication $\Delta(g) = g \otimes g$ and counit $\varepsilon: kG \rightarrow k$. Let $I = \ker \varepsilon$.

(Quillen 1968) The $I$-adic completion of the group-algebra, $\hat{kG} = \lim_{\leftarrow k} kG/I^k$, is a filtered, complete Hopf algebra.

An element $x \in \hat{kG}$ is called primitive if $\hat{\Delta}x = x \hat{\otimes} 1 + 1 \hat{\otimes} x$. The set of all such elements, with bracket $[x, y] = xy - yx$, and endowed with the induced filtration, is a complete, filtered Lie algebra.

We then have $m(G) \cong \text{Prim}(\hat{kG})$ and $\text{gr}(m(G)) \cong \text{gr}(G)$.

(Sullivan 1977) $G$ is 1-formal $\iff m(G)$ is quadratic, namely:

$$m(G) = \mathfrak{h}(\widehat{H^\bullet(G)}, k).$$

ALEX SUCIU (NORTHEASTERN)
Theorem (PS 2017)

A f.g. group $G$ admits a $1$-finite $1$-model $A$ if and only if $m(G)$ is the lcs completion of a finitely presented Lie algebra, namely,

$$m(G) \cong \hat{h}(A).$$

Theorem (PS 2017)

Let $G$ be a f.g. group which has a free, non-cyclic quotient. Then:

- $G/G''$ is not finitely presentable.
- $G/G''$ does not admit a $1$-finite $1$-model.