Finiteness and formality obstructions

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Let $A = (A^\bullet, d)$ be a commutative, differential graded algebra over a field $\mathbb{k}$ of characteristic 0. That is:

- $A = \bigoplus_{i \geq 0} A^i$, where $A^i$ are $\mathbb{k}$-vector spaces.
- The multiplication $\cdot : A^i \otimes A^j \to A^{i+j}$ is graded-commutative, i.e., $ab = (-1)^{|a||b|} ba$ for all homogeneous $a$ and $b$.
- The differential $d : A^i \to A^{i+1}$ satisfies the graded Leibnitz rule, i.e., $d(ab) = d(a)b + (-1)^{|a|} a d(b)$.

We assume $A$ is connected (i.e., $A^0 = \mathbb{k} \cdot 1$) and of finite-type (i.e., $\dim A^i < \infty$ for all $i$).

For each $a \in Z^1(A) \cong H^1(A)$, we have a cochain complex,

$$
(A^\bullet, \delta_a) : \quad A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,
$$

with differentials $\delta_a^i(u) = a \cdot u + d(u)$, for all $u \in A^i$.

The **resonance varieties** of $A$ are the affine varieties

$$
\mathcal{R}_s^i(A) = \{ a \in H^1(A) \mid \dim_{\mathbb{k}} H^i(A^\bullet, \delta_a) \geq s \}.
$$
Fix a $k$-basis $\{e_1, \ldots, e_r\}$ for $A^1$, and let $\{x_1, \ldots, x_r\}$ be the dual basis for $A_1 = (A^1)^*$. 

Identify $\text{Sym}(A_1)$ with $S = k[x_1, \ldots, x_r]$, the coordinate ring of the affine space $A^1$.

Build a cochain complex of free $S$-modules, $L(A) := (A^* \otimes S, \delta)$:

\[ \cdots \longrightarrow A^i \otimes S \xrightarrow{\delta^i} A^{i+1} \otimes S \xrightarrow{\delta^{i+1}} A^{i+2} \otimes S \longrightarrow \cdots, \]

where $\delta^i(u \otimes f) = \sum_{j=1}^r e_j u \otimes f x_j + d u \otimes f$.

The specialization of $(A \otimes S, \delta)$ at $a \in Z^1(A)$ is $(A, \delta_a)$.

Hence, $R^i_s(A)$ is the zero-set of the ideal generated by all minors of size $b_i(A) - s + 1$ of the block-matrix $\delta^{i+1} \oplus \delta^i$. 
Let $X$ be a connected, finite-type CW-complex. Then $\pi = \pi_1(X, x_0)$ is a finitely presented group, with $\pi_{ab} \cong H_1(X, \mathbb{Z})$.

The ring $R = \mathbb{C}[\pi_{ab}]$ is the coordinate ring of the character group, $\text{Char}(X) = \text{Hom}(\pi, \mathbb{C}^*) \cong (\mathbb{C}^*)^r \times \text{Tors}(\pi_{ab})$, where $r = b_1(X)$.

The **characteristic varieties** of $X$ are the homology jump loci

$$\mathcal{V}_s^i(X) = \{\rho \in \text{Char}(X) | \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geq s\}.$$ 

These varieties are homotopy-type invariants of $X$, with $\mathcal{V}_s^1(X)$ depending only on $\pi = \pi_1(X)$.

Set $\mathcal{V}_1^1(\pi) := \mathcal{V}_1^1(K(\pi, 1))$; then $\mathcal{V}_1^1(\pi) = \mathcal{V}_1(\pi/\pi'')$.

**Example**

Let $f \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ be a Laurent polynomial, $f(1) = 0$. There is then a finitely presented group $\pi$ with $\pi_{ab} = \pi^n$ such that $\mathcal{V}_1^{1}(\pi) = \mathcal{V}(f)$. 

**ALEX SUCIU (Northeastern)**

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**Tangent cones**

- Let \( \exp : H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^*) \) be the coefficient homomorphism induced by \( \mathbb{C} \to \mathbb{C}^*, \ z \mapsto e^z \).

- Let \( W = V(I) \), a Zariski closed subset of \( \text{Char}(G) = H^1(X, \mathbb{C}^*) \).

- The *tangent cone* at \( 1 \) to \( W \) is \( TC_1(W) = V(\text{in}(I)) \).

- The *exponential tangent cone* at \( 1 \) to \( W \):

  \[ \tau_1(W) = \{ z \in H^1(X, \mathbb{C}) \mid \exp(\lambda z) \in W, \ \forall \lambda \in \mathbb{C} \} \]

  Both tangent cones are homogeneous subvarieties of \( H^1(X, \mathbb{C}) \); are non-empty iff \( 1 \in W \); depend only on the analytic germ of \( W \) at \( 1 \); commute with finite unions and arbitrary intersections.

- \( \tau_1(W) \subseteq TC_1(W) \), with = if all irred components of \( W \) are subtori, but \( \neq \) in general.

- (Dimca–Papadima–S. 2009) \( \tau_1(W) \) is a finite union of rationally defined subspaces.
Algebraic models for spaces

- A CDGA map $\varphi: A \to B$ is a quasi-isomorphism if $\varphi^*: H^\bullet(A) \to H^\bullet(B)$ is an isomorphism.

- $\varphi$ is a $q$-quasi-isomorphism (for some $q \geq 1$) if $\varphi^*$ is an isomorphism in degrees $\leq q$ and is injective in degree $q + 1$.

- Two CDGAs, $A$ and $B$, are ($q$-) equivalent if there is a zig-zag of ($q$-) quasi-isomorphisms connecting $A$ to $B$.

- $A$ is formal (or just $q$-formal) if it is ($q$-) equivalent to $(H^\bullet(A), d = 0)$.

- A CDGA is $q$-minimal if it is of the form $(\bigwedge V, d)$, where the differential structure is the inductive limit of a sequence of Hirsch extensions of increasing degrees, and $V^i = 0$ for $i > q$.

- Every CDGA $A$ with $H^0(A) = \mathbb{k}$ admits a $q$-minimal model, $M_q(A)$ (i.e., a $q$-equivalence $M_q(A) \to A$ with $M_q(A) = (\bigwedge V, d)$ a $q$-minimal cdga), unique up to iso.
Given any (path-connected) space $X$, there is an associated Sullivan $\mathbb{Q}$-cdga, $A_{PL}(X)$, such that $H^\bullet(A_{PL}(X)) = H^\bullet(X, \mathbb{Q})$.

An algebraic $(q)$-model (over $k$) for $X$ is a $k$-cgda $(A, d)$ which is $(q)$-equivalent to $A_{PL}(X) \otimes_{\mathbb{Q}} k$.

If $M$ is a smooth manifold, then $\Omega_{dR}(M)$ is a model for $M$ (over $\mathbb{R}$).

Examples of spaces having finite-type models include:

- Formal spaces (such as compact Kähler manifolds, hyperplane arrangement complements, toric spaces, etc).
- Smooth quasi-projective varieties, compact solvmanifolds, Sasakian manifolds, etc.
Let $X$ be a connected CW-complex with finite $q$-skeleton. Suppose $X$ admits a $q$-finite $q$-model $A$.

**Theorem**

For all $i \leq q$ and all $s$:

- (DPS 2009, Dimca–Papadima 2014) $V^i_s(X)_1 \cong R^i_s(A)_0$.
- (Budur–Wang 2017) All the irreducible components of $V^i_s(X)$ passing through the origin of $\text{Char}(X)$ are algebraic subtori.

Consequently,

$$\tau_1(V^i_s(X)) = TC_1(V^i_s(X)) = R^i_s(A).$$

**Theorem (Papadima–S. 2017)**

A f.g. group $G$ admits a 1-finite 1-model if and only if the Malcev Lie algebra $m(G)$ is the LCS completion of a finitely presented Lie algebra.
**Theorem**

Let $X$ be a connected CW-complex with finite $q$-skeleton. Suppose $X$ admits a $q$-finite $q$-model $A$. Then, for all $i \leq q$ and all $s$,

- *(Dimca–Papadima 2014)* \( V^i_s(X)_{(1)} \cong R^i_s(A)_{(0)} \).
  
  In particular, if $X$ is $q$-formal, then \( V^i_s(X)_{(1)} \cong R^i_s(X)_{(0)} \).

- *(Macinic, Papadima, Popescu, S. 2017)* \( TC_0(R^i_s(A)) \subseteq R^i_s(X) \).

- *(Budur–Wang 2017)* All the irreducible components of $V^i_s(X)$ passing through the origin of $H^1(X, \mathbb{C}^*)$ are algebraic subtori.

**Example**

Let $G$ be a f.p. group with $G_{ab} = \mathbb{Z}^n$ and $V^1_1(G) = \{ t \in (\mathbb{C}^*)^n \mid \sum_{i=1}^n t_i = n \}$. Then $G$ admits no 1-finite 1-model.
**Theorem (Papadima–S. 2017)**

Suppose $X$ is $(q + 1)$ finite, or $X$ admits a $q$-finite $q$-model. Then $b_i(\mathcal{M}_q(X)) < \infty$, for all $i \leq q + 1$.

**Corollary**

Let $G$ be a f.g. group. Assume that either $G$ is finitely presented, or $G$ has a 1-finite 1-model. Then $b_2(\mathcal{M}_1(G)) < \infty$.

**Example**

- Consider the free metabelian group $G = F_n / F''_n$ with $n \geq 2$.
- We have $\mathcal{V}^1(G) = \mathcal{V}^1(F_n) = (\mathbb{C}^*)^n$, and so $G$ passes the Budur–Wang test.
- But $b_2(\mathcal{M}_1(G)) = \infty$, and so $G$ admits no 1-finite 1-model (and is not finitely presented).
Let $G$ be a group. The *lower central series* $\{\gamma_k(G)\}_{k \geq 1}$ is defined inductively by $\gamma_1(G) = G$ and $\gamma_{k+1}(G) = [G, \gamma_k(G)]$.

Here, if $H, K < G$, then $[H, K]$ is the subgroup of $G$ generated by $\{[a, b] := aba^{-1}b^{-1} \mid a \in H, b \in K\}$. If $H, K \triangleleft G$, then $[H, K] \triangleleft G$.

The subgroups $\gamma_k(G)$ are, in fact, characteristic subgroups of $G$. Moreover $[\gamma_k(G), \gamma_\ell(G)] \subseteq \gamma_{k+\ell}(G)$, $\forall k, \ell \geq 1$.

$\gamma_2(G) = [G, G]$ is the derived subgroup, and so $G/\gamma_2(G) = G_{ab}$.

$[\gamma_k(G), \gamma_k(G)] \triangleleft \gamma_{k+1}(G)$, and thus the LCS quotients, 

$$\text{gr}_k(G) := \gamma_k(G)/\gamma_{k+1}(G)$$

are abelian.

If $G$ is finitely generated, then so are its LCS quotients. Set $\phi_k(G) := \text{rank } \text{gr}_k(G)$. 
**Associated graded Lie algebra**

- Fix a coefficient ring \( \mathbb{k} \). Given a group \( G \), we let

\[
\text{gr}(G, \mathbb{k}) = \bigoplus_{k \geq 1} \text{gr}_k(G) \otimes \mathbb{k}.
\]

- This is a graded Lie algebra, with Lie bracket \([, :] : \text{gr}_k \times \text{gr}_\ell \to \text{gr}_{k+\ell}\) induced by the group commutator.

- For \( \mathbb{k} = \mathbb{Z} \), we simply write \( \text{gr}(G) = \text{gr}(G, \mathbb{Z}) \).

- The construction is functorial.

- Example: if \( F_n \) is the free group of rank \( n \), then
  
  - \( \text{gr}(F_n) \) is the free Lie algebra \( \text{Lie}(\mathbb{Z}^n) \).
  - \( \text{gr}_k(F_n) \) is free abelian, of rank \( \phi_k(F_n) = \frac{1}{k} \sum_{d \mid k} \mu(d) n^{k \over d} \). 

A quadratic approximation of the Lie algebra $\text{gr}(G, k)$, where $k$ is a field, is the *holonomy Lie algebra* of $G$, which is defined as

$$\mathfrak{h}(G, k) := \text{Lie}(H_1(G, k))/\langle \text{im}(\mu^\vee_G) \rangle,$$

where

- $L = \text{Lie}(V)$ the free Lie algebra on the $k$-vector space $V = H_1(G; k)$, with $L_1 = V$ and $L_2 = V \wedge V$.
- $\mu^\vee_G : H_2(G, k) \rightarrow V \wedge V$ is the dual of the cup product map $\mu_G : H^1(G; k) \wedge H^1(G; k) \rightarrow H^2(G; k)$.

There is a surjective morphism of graded Lie algebras,

$$\mathfrak{h}(G, k) \twoheadrightarrow \text{gr}(G; k),$$

which restricts to isomorphisms $\mathfrak{h}_k(G, k) \rightarrow \text{gr}_k(G; k)$ for $k \leq 2$. 
Let $\mathcal{A} = \{\ell_1, \ldots, \ell_n\}$ be an affine line arrangement in $\mathbb{C}^2$, and let $G = G(\mathcal{A})$ be the fundamental group of the complement of $\mathcal{A}$.

The holonomy Lie algebra $\mathfrak{h}(\mathcal{A}) := \mathfrak{h}(G(\mathcal{A}))$ has (combinatorially determined) presentation

$$\mathfrak{h}(\mathcal{A}) = \langle x_1, \ldots, x_n \mid \sum_{k \in \hat{P}} [x_j, x_k], \ j \in \hat{P}, \ P \in \mathcal{P} \rangle$$

where $x_i$ represents the meridian about the $i$-th line, $\mathcal{P} \subset 2^n$ is the set of multiple points, and $\hat{P} = P \setminus \{\max P\}$ for $P \in \mathcal{P}$.

Thus, every double point $P = L_i \cap L_j$ contributes a relation $[x_i, x_j]$, each triple point $P = L_i \cap L_j \cap L_k$ contributes two relations, $[x_i, x_j] + [x_i, x_k]$ and $-[x_i, x_j] + [x_j, x_k]$, etc.

Consequently, $\mathfrak{h}_1(\mathcal{A})$ is free abelian with basis $\{x_1, \ldots, x_n\}$, while $\mathfrak{h}_2(\mathcal{A})$ is free abelian of rank $\phi_2 = \binom{n}{2} - \sum_{P \in \mathcal{P}}(|P| - 1)$, with basis $\{[x_i, x_j] : i, j \in \hat{P}, \ P \in \mathcal{P}\}$.
The canonical projection $\mathfrak{h}(G, \mathbb{Q}) \to \operatorname{gr}(G, \mathbb{Q})$ is an isomorphism. Thus, the LCS ranks $\phi_k(G)$ are combinatorially determined.

(Falk–Randell 1985) If $\mathcal{A}$ is supersolvable, with exponents $d_1, \ldots, d_\ell$, then $G = F_{d_\ell} \times \cdots \times F_{d_2} \times F_{d_1}$ (almost direct product) and
\[
\phi_k(G) = \sum_{i=1}^{\ell} \phi_k(F_{d_i}).
\]

(Papadima–S. 2006) If $\mathcal{A}$ is decomposable, then $\mathfrak{h}(G) \to \operatorname{gr}(G)$ is an isomorphism, and $\operatorname{gr}_k(G)$ is free abelian of rank
\[
\phi_k(G) = \sum_{\chi \in \mathcal{L}_2(\mathcal{A})} \phi_k(F_{\mu(\chi)}) \quad \text{for} \ k \geq 2.
\]

(S. 2001) For $G = G(\mathcal{A})$, the groups $\operatorname{gr}_k(G)$ may have non-zero torsion. Question: Is that torsion combinatorially determined?

(Artal Bartolo, Guerville-Ballé, and Viu-Sos 2018): Answer: No!
Malcev Lie algebra

- Let $k$ be a field of characteristic 0. The group-algebra $kG$ has a natural Hopf algebra structure, with comultiplication $\Delta(g) = g \otimes g$ and counit $\varepsilon : kG \to k$.

- Let $I = \ker \varepsilon$. The $I$-adic completion $\hat{k}G = \varprojlim_k kG/I^k$ is a filtered, complete Hopf algebra.

- An element $x \in \hat{k}G$ is called primitive if $\hat{\Delta}x = x\hat{\otimes}1 + 1\hat{\otimes}x$. The set of all such elements,

$$m(G, k) = \text{Prim}(\hat{k}G),$$

with bracket $[x, y] = xy - yx$, is a complete, filtered Lie algebra, called the Malcev Lie algebra of $G$.

- If $G$ is finitely generated, then $m(G, k) = \varprojlim_k \mathcal{L}(G/\gamma_k(G) \otimes k)$, and

$$\text{gr}(m(G, k)) \cong \text{gr}(G, k).$$
Formality properties

Formality and filtered formality

Let $G$ be a finitely generated group, $k$ a field of characteristic 0.

$G$ is \textit{filtered-formal} (over $k$), if there is an isomorphism of filtered Lie algebras,

$$m(G; k) \cong \mathfrak{gr}(G; k).$$

$G$ is \textit{1-formal} (over $k$) if it is filtered formal and the canonical projection $\mathfrak{h}(G, k) \rightarrow \mathfrak{gr}(G; k)$ is an isomorphism; that is,

$$m(G; k) \cong \mathfrak{h}(G; k).$$

An obstruction to 1-formality is provided by the Massey products

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle \in H^2(G, k), \text{ for } \alpha_i \in H^1(G, k) \text{ with } \alpha_1 \alpha_2 = \alpha_2 \alpha_3 = 0.$$

\textbf{Theorem (S.–Wang)}

The above formality properties are preserved under finite direct products and coproducts, split injections, passing to solvable quotients, as well as extension or restriction of coefficient fields.
Examples of 1-formal groups

- Fundamental groups of compact Kähler manifolds; e.g., surface groups.
- Fundamental groups of complements of complex algebraic affine hypersurfaces; e.g., arrangement groups, free groups.
- Right-angled Artin groups.

Examples of filtered formal groups

- Finitely generated, torsion-free, 2-step nilpotent groups with torsion-free abelianization; e.g., the Heisenberg group.
- Fundamental groups of Sasakian manifolds.
- Fundamental groups of graphic configuration spaces of surfaces of genus $g \geq 1$; e.g., pure braid groups of elliptic curves.

Examples of non-filtered formal groups

- Certain finitely generated, torsion-free, 3-step nilpotent groups.
Consider the tower of nilpotent quotients of a group $G$,

$$
\cdots \rightarrow G/\gamma_4(G) \overset{q_3}{\rightarrow} G/\gamma_3(G) \overset{q_2}{\rightarrow} G/\gamma_2(G) .
$$

We then have central extensions

$$
0 \rightarrow \text{gr}_k(G) \rightarrow G/\gamma_{k+1}(G) \overset{q_k}{\rightarrow} G/\gamma_k(G) \rightarrow 0 .
$$

Passing to classifying spaces, we obtain commutative diagrams,

$$
\begin{array}{c}
G \\
\downarrow \psi_k \\
K(G/\gamma_k(G), 1)
\end{array} \xrightarrow{\psi_{k+1}} \begin{array}{c}
G/\gamma_{k+1}(G) \\
\downarrow \pi_k \\
K(G/\gamma_{k+1}(G), 1)
\end{array}
$$

The map $\pi_k$ may be viewed as the fibration with fiber $K(\text{gr}_k(G), 1)$ obtained as the pullback of the path space fibration with base $K(\text{gr}_k(G), 2)$ via a $k$-invariant $\chi_k : K(G/\gamma_k(G), 1) \rightarrow K(\text{gr}_k(G), 2)$. 
Let $X$ be a connected CW-complex, and let $G = \pi_1(X)$.

A $K(G, 1)$ can be constructed by adding to $X$ cells of dimension 3 or higher. Thus, $H_2(G, \mathbb{Z})$ is a quotient of $H_2(X, \mathbb{Z})$.

Let $\iota: X \to K(G, 1)$ be the inclusion, and let

$$h_k = \psi_k \circ \iota: X \to K(G/\gamma_k(G), 1).$$

We obtain a Postnikov tower of fibrations,
As noted by Stallings, there is an exact sequence,

$$H_2(X; \mathbb{Z}) \xrightarrow{(h_k)_*} H_2(G/\gamma_k(G); \mathbb{Z}) \xrightarrow{\chi_k} \text{gr}_k(G) \longrightarrow 0.$$  

In general, this sequence is natural but not split exact.

The homomorphism

$$(h_2)_*: H_2(X; \mathbb{Z}) \longrightarrow H_2(G/\gamma_2(G); \mathbb{Z}) \cong H_1(G; \mathbb{Z}) \wedge H_1(G; \mathbb{Z})$$

is the *holonomy map* of $X$ (over $\mathbb{Z}$).

When $H_1(G; \mathbb{Z})$ is torsion-free, set

$$\mathfrak{h}(G) = \text{Lie}(H_1(G; \mathbb{Z}))/\langle \text{im}((h_2)_*) \rangle.$$  

As before, get surjective morphism $\mathfrak{h}(G) \twoheadrightarrow \text{gr}(G)$, which is injective in degrees $k \leq 2$. 
Suppose $H = H_1(G; \mathbb{Z})$ is a finitely-generated, free abelian group, and the map $(h_2)_*: H_2(G; \mathbb{Z}) \to H \wedge H$ is injective.

**Theorem (Rybnikov, Porter–S.)**

The canonical projection $h_3(G) \to \text{gr}_3(G)$ is an isomorphism.

**Theorem (Porter–S.)**

For each $k \geq 3$, there is a split exact sequence,

\[ 0 \to \text{gr}_k(G) \xrightarrow{i} H_2(G/\gamma_k(G); \mathbb{Z}) \xrightarrow{\pi} H_2(X; \mathbb{Z}) \to 0. \quad (\dagger) \]

Moreover, the $k$-invariant of the extension from $G/\gamma_k(G)$ to $G/\gamma_{k+1}(G)$,

\[ \chi_k \in \text{Hom}(H_2(G/\gamma_k(G)), \text{gr}_k(G)), \]

with respect to the direct sum decomposition defined by $\sigma$, is given by $\chi_k(x, c) = x - \lambda(c)$, where $\lambda = \sigma \circ (h_k)_*$. 
Let $X_a$ and $X_b$ be two path-connected spaces with
- Finitely generated, torsion-free $H_1$.
- Injective holonomy map $H_2 \to H_1 \wedge H_1$.

Let $G_a$ and $G_b$ be the respective fundamental groups.

A homomorphism $f: G_a \to G_b$ induces homomorphisms on nilpotent quotients, $f_k: G_a/\gamma_k(G_a) \to G_b/\gamma_k(G_b)$.

Suppose there is an isomorphism of graded algebras,

$$g: H^{\leq 2}(X_b) \to H^{\leq 2}(X_a).$$

Set $\overline{g} = g^\vee: H_{\leq 2}(X_a) \to H_{\leq 2}(X_b)$.

There is then an isomorphism $G_a/\gamma_3(G_a) \cong G_b/\gamma_3(G_b)$.

Moreover, the isomorphism $\overline{g}_1: H_1(X_a) \to H_1(X_b)$ induces an isomorphism $\overline{g}_\# : \mathfrak{h}_3(G_a) \to \mathfrak{h}_3(G_b)$. 
Theorem (Rybnikov, Porter–S.)

Let $\sigma_b : H_2(G_b/\Gamma_3(G_b)) \to \eta_3(G_b)$ be any left splitting of $(\dagger)$, and let $f_3 : G_a/\gamma_3(G_a) \xrightarrow{\cong} G_b/\gamma_3(G_b)$ be any extension of $\bar{g}$. Then $f_3$ extends to an isomorphism

$$f_4 : G_a/\gamma_4(G_a) \xrightarrow{\cong} G_b/\gamma_4(G_b)$$

if and only if there are liftings $h^c_3 : X_c \to K(G_c/\gamma_3(G_c), 1)$ for $c = a$ and $b$ such that the following diagram commutes
Let $p = 0$ or a prime.

Given a group $G$, define subgroups $\gamma_k^p(G)$ as $\gamma_1^p(G) = G$ and

$$\gamma_{k+1}^p(G) = \langle gug^{-1}u^{-1}v^p : g \in G, u, v \in \gamma_k^p(G) \rangle.$$ 

$\{\gamma_k^p(G)\}_{k \geq 1}$ is a descending central series of normal subgroups.

For $p = 0$ it is the LCS; for $p \neq 0$ it is the most rapidly descending central series whose successive quotients are $\mathbb{Z}_p$-vector spaces.

All the above results work for $p > 0$, by replacing $\gamma_k(G) \sim \gamma_k^p(G)$, $\mathfrak{h}_k(G) \sim \mathfrak{h}_k(G, \mathbb{Z}_p)$, and $H_*(-, \mathbb{Z}) \sim H_*(-, \mathbb{Z}_p)$.

The entries of the matrices $\lambda_a$ and $\lambda_b$ are generalized Massey triple products in $H^2(X_b, \mathbb{Z}_p)$ and $H^2(X_a, \mathbb{Z}_p)$, respectively.
Rybnikov’s arrangements

- For groups of hyperplane arrangements, $h_2$ and $h_3$ are torsion free. Moreover, the holonomy map is injective, and so $h_3 \cong \text{gr}_3$.

- The obstruction to extending $\bar{g}$ to an isomorphism from $G/\gamma_4(G_a)$ to $G/\gamma_4(G_b)$ is computed by generalized Massey triple products.

- Rybnikov used the above theorem (with $n = 3$) to show that arrangement groups are not combinatorially determined.

- Starting from a realization $A$ of the MacLane matroid over $\mathbb{C}$, he constructed a pair of arrangements of 13 planes in $\mathbb{C}^3$, $A^+$ and $A^-$, such that
  - $L(A^+) \cong L(A^-)$, and thus $G^+ / \gamma_3(G^+) \cong G^- / \gamma_3(G^-)$.
  - $G^+ / \gamma_4(G^+) \not\cong G^- / \gamma_4(G^-)$.

- Goal: Make explicit the generalized Massey products (over $\mathbb{Z}_3$) that distinguish these two nilpotent quotients.
