RECOGNIZING SURFACES

Ivo Nikolov

and

Alexandru I. Suciu

Mathematics Department

College of Arts and Sciences

Northeastern University

Abstract

The subject of this poster is the interplay between the topology and the combinatorics of surfaces. The main problem of Topology is to classify spaces up to continuous deformations, known as homeomorphisms. Under certain conditions, topological invariants that capture qualitative and quantitative properties of spaces lead to the enumeration of homeomorphism types.

Surfaces are some of the simplest, yet most interesting topological objects. The poster focuses on the main topological invariants of two-dimensional manifolds—orientability, number of boundary components, genus, and Euler characteristic—and how these invariants solve the classification problem for compact surfaces.

The poster introduces a Java applet that was written in Fall, 1998 as a class project for a Topology I course. It implements an algorithm that determines the homeomorphism type of a closed surface from a combinatorial description as a polygon with edges identified in pairs. The input for the applet is a string of integers, encoding the edge identifications. The output of the applet consists of three topological invariants that completely classify the resulting surface.

Topology of Surfaces

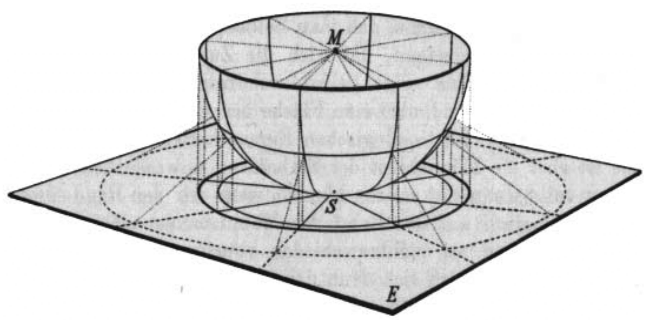
Topology is the abstraction of certain geometrical ideas, such as continuity and closeness. Roughly speaking, topology is the exploration of manifolds, and of the properties that remain invariant under continuous, invertible transformations, known as homeomorphisms. The basic problem is to classify manifolds according to homeomorphism type. In higher dimensions, this is an impossible task, but, in low dimensions, it can be done.

Surfaces are some of the simplest, yet most interesting topological objects. They are *compact* and *connected* spaces with the following property: each point has a neighborhood homeomorphic to either

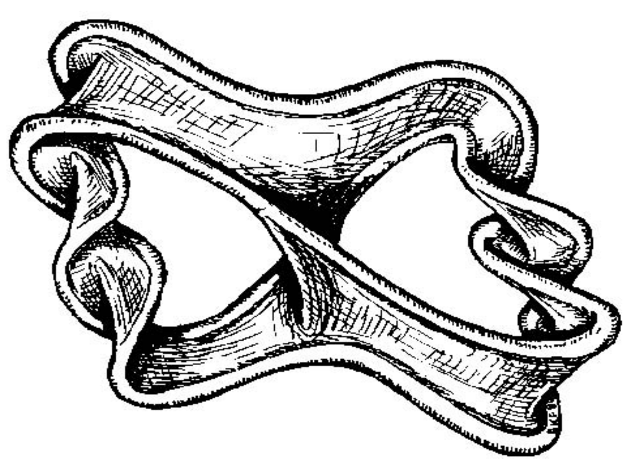
- the plane $\mathbf{R}^2$, or
- the half-plane $\mathbf{H}^2$.

Points of the first type are called *interior* points, and those of the second type are called *boundary* points. The set of all boundary points constitutes the boundary of the surface. It consists of one or *boundary components,* each of which is homeomorphic to a circle.

If the surface has no boundary, it is called a *closed* surface. For example, the sphere $\mathbf{S}^2$ and the torus $\mathbf{T}^2$ are closed surfaces. The disk has one boundary curve (a circle), and is topologically the same as a hemisphere (a sphere with a disk removed):



The surface below is a torus with a disk removed:



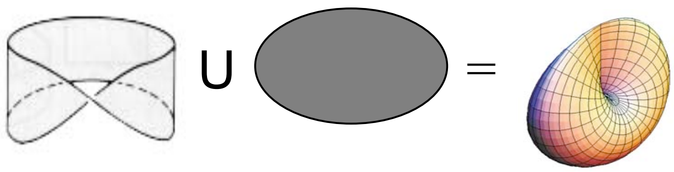
Closed-up surfaces

The classification of all surfaces essentially reduces to that of closed surfaces. To see why this is the case, consider an arbitrary surface **S**. To each boundary component (which, recall, is nothing but a circle), attach a disk. The resulting space, call it $\mathbf{S}^\wedge$ (the *closed-up* **S**) is clearly a closed surface. The closing-up operation preserves homeomorphism types, i.e.:

$$\mathbf{S}_1 \approx \mathbf{S}_2 \text{ if and only if } \mathbf{S}^\wedge_1 \approx \mathbf{S}^\wedge_2$$

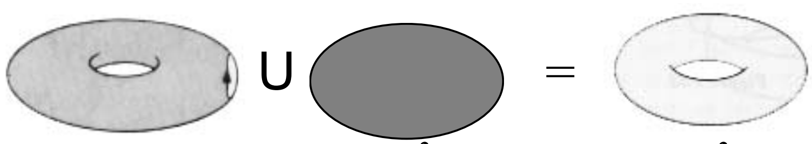
Thus, can divide surfaces into classes, where two surfaces are in the same class if they have homeomorphic closed-up surfaces.

Examples:



$\mathbf{Mb} \cup \mathbf{D}^2 = \mathbf{RP}^2$

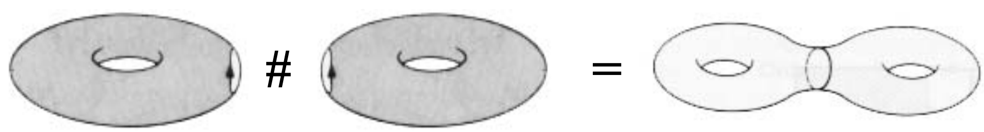
When we attach a disk to the boundary of the Moebius Strip we get the Projective Plane, or Crosscap



Punctured torus $\cup$ $\mathbf{D}^2 = \mathbf{T}^2$

Connected sums

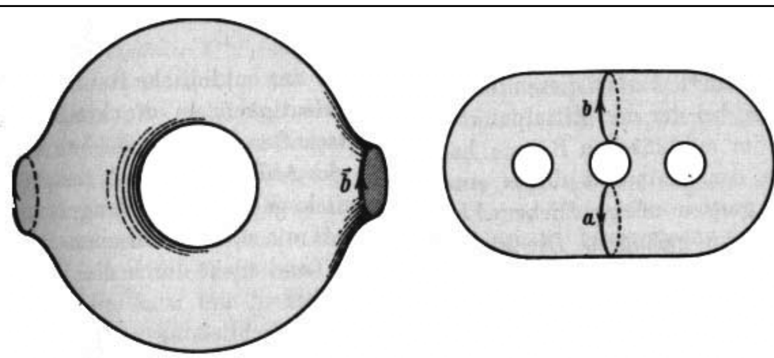
Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two closed surfaces. Cut out a disk from each one, and attach the two resulting surfaces along their boundary. The result is a closed surface, $\mathbf{S}_1 \# \mathbf{S}_2$, called the *connected sum* of the two surfaces.



This picture can give an idea to the reader. When we connect two tori, we get a double torus.

It can be shown that connected sum does not depend on the choice of disks that are cut out from each surface, and so it is a well-defined operation. Moreover, the connected sum operation respects homeomorphisms:

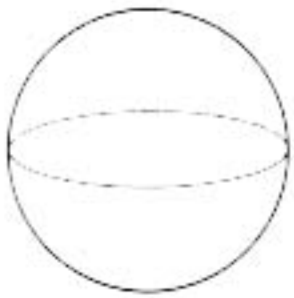
$$\text{If } \mathbf{S}_1 \approx \mathbf{S}'_1 \text{ and } \mathbf{S}_2 \approx \mathbf{S}'_2 \text{ then } \mathbf{S}_1 \# \mathbf{S}_2 \approx \mathbf{S}'_1 \# \mathbf{S}'_2$$



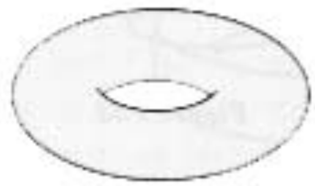
If we take a torus, cut two disks from it and then attach two such twice-punctured tori, we get the triple torus.

Some Basic Surfaces

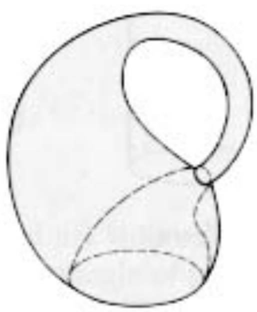
This is where all begins and we introduce the most general surfaces.



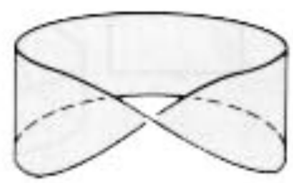
The Sphere $\mathbf{S}^2$



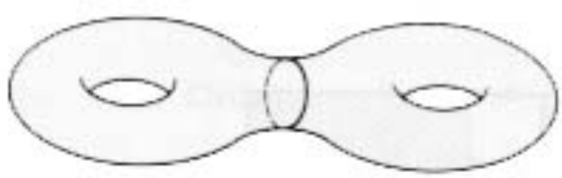
The Torus $\mathbf{T}^2$



The Klein bottle $\mathbf{K}^2$



Moebius band **Mb**



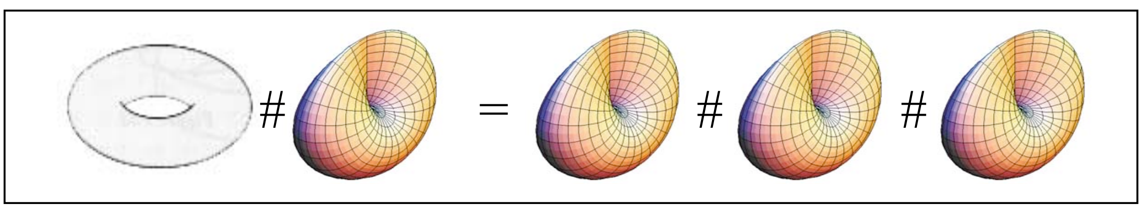
The Double torus

Classification of Surfaces

The *Main Classification Theorem* for surfaces states that every closed surface is homeomorphic to a sphere with some “handles” or “crosscaps” attached. That is, every single surface is one of the following:

- $\mathbf{S}^2$
- $\mathbf{RP}^2 \# \mathbf{RP}^2 \# \dots \# \mathbf{RP}^2$
- $\mathbf{T}^2 \# \mathbf{T}^2 \# \dots \# \mathbf{T}^2$

One can ask what happens if we attach a handle and a crosscap to a sphere. The answer can be found in the following fact: $\mathbf{RP}^2 \# \mathbf{T}^2$ is homeomorphic to $\mathbf{RP}^2 \# \mathbf{RP}^2 \# \mathbf{RP}^2$.



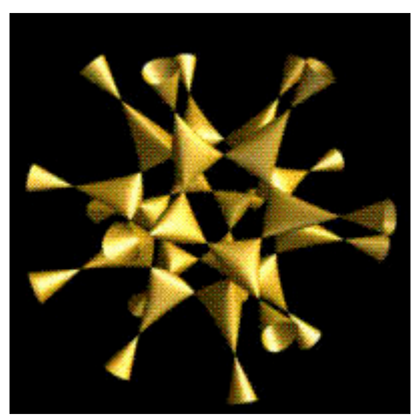
Invariants of Surfaces

In order to better understand surfaces, we need some simple characteristics that capture their essential qualitative and qualitative properties. Such characteristics should remain the same for homeomorphic surfaces—that is why they are called (topological) *invariants*. It turns out that only three invariants are needed for the complete classification of surfaces.

- **Number of boundary components.**

This is an integer **c,** counting the number of boundary components of the surface.

Can you tell how many boundaries these surfaces have?

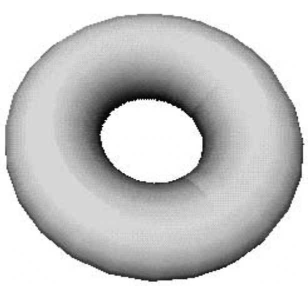


- **Orientability.**

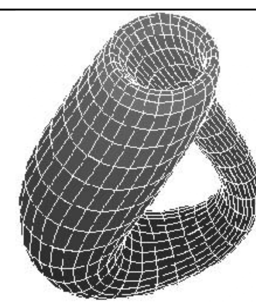
This is a boolean value ε. To understand it, let us consider a closed curve in the surface, homeomorphic to a circle. Each of its closed neighborhoods in the surface is homeomorphic to a cylinder or a Moebius Strip, depending on the parity of the number of twists in it. A surface is called *orientable* if all of these are cylinders (ε=1), and *non-orientable* if there is at least one Moebius Strip (ε=0).

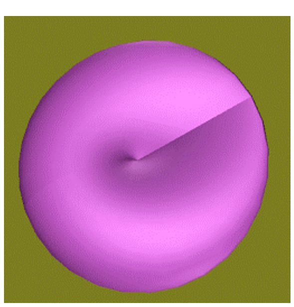
Examples:

The 1st, the 3rd and the 4th surfaces are orientable, while the 2nd is non-orientable – it has just one side of the band

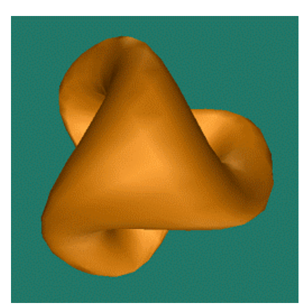


The torus (on the left) is an orientable surface, while the Klein bottle (on the right) is not, since it does not enclose any space, even though it is closed





The real projective plane is non-orientable surface that cannot be realized in $\mathbf{R}^3$. It is essentially the same as the set of all lines, passing through a given point in $\mathbf{R}^3$.

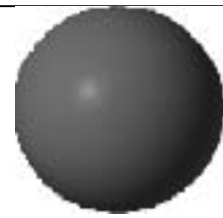


- **Genus.**

This is an integer **g** that counts the number of handles (if **ε** = 1) or crosscaps (if **ε** = 0) in a closed surface.

Examples:

The torus is a closed surface of genus 1.



The sphere is a closed surface of genus 0.

We also set the genus of a surface with boundary to be equal to the corresponding closed surface. For example, the genus of a disk is the same as that of a sphere, namely *0*. The same is true for the annulus. The genus of the Moebius band is the same as that of the projective space, which is *1*.

- **Euler Characteristic**

Besides the above three invariants, there is another general invariant of spaces: the Euler characteristic, χ. For a polyhedron, this is given by

$$\chi = \mathbf{v} - \mathbf{e} + \mathbf{f}$$

where

- **v** is the number of vertices
- **e** is the number of edges
- **f** is the number of faces

For a surface, it turns out that the Euler characteristic can be expressed solely in terms of the three invariants above. Namely:

$$\chi = 2 - 2\mathbf{g} - \mathbf{c} \qquad \text{if } \varepsilon = 1$$

$$\chi = 2 - \mathbf{g} - \mathbf{c} \qquad \text{if } \varepsilon = 0$$

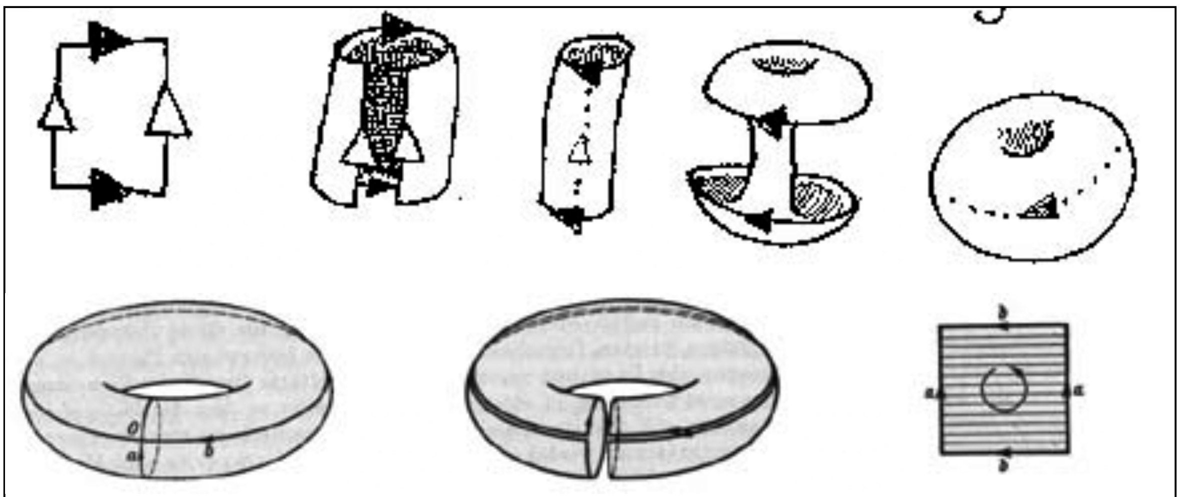
For example, if we take the sphere—a closed orientable surface of genus *0*—the Euler characteristic is *2*, according to the latter formula. Now, consider an empty cube. It is homeomorphic to the sphere, it has *8* vertices, *12* edges and *6* sides—so, the Euler characteristic is *2* according to the first formula, also.

Examples

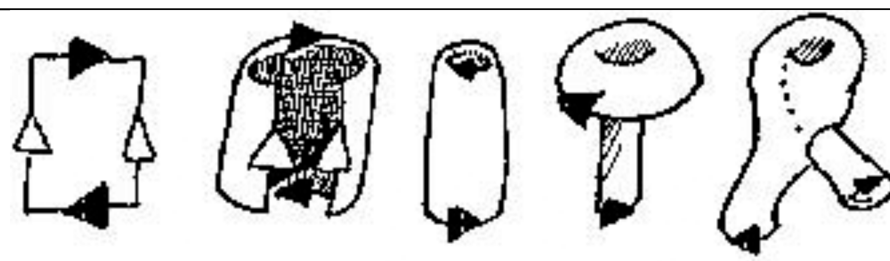
Surface	**g**	**ε**	**c**	**χ**
Disk	*0*	*1*	*0*	*1*
Sphere	*0*	*1*	*1*	*2*
Annulus	*0*	*1*	*2*	*0*
Moebius band	*1*	*0*	*1*	*0*
Projective space	*1*	*0*	*0*	*1*
Torus	*1*	*1*	*0*	*0*
Klein bottle	*2*	*0*	*0*	*0*
Double torus	*2*	*1*	*0*	*–2*
Punctured torus	*2*	*1*	*1*	*–1*

Surfaces as Polygons with Sides Identified

One way to understand surfaces is to view them as polygons with sides identified according to some specific, purely combinatorial rules. The polygon lies in the real plane and the nice thing is that we can represent each closed surface this way. We identify each if its sides to another one and keep track of the direction we do this. That is how we do it:

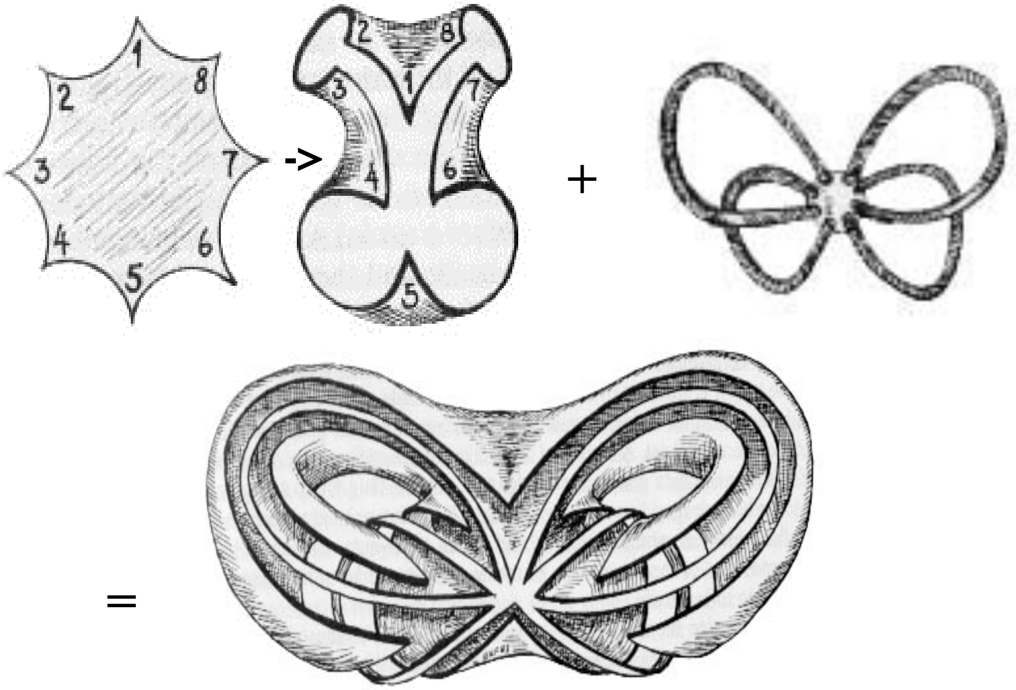


For the torus

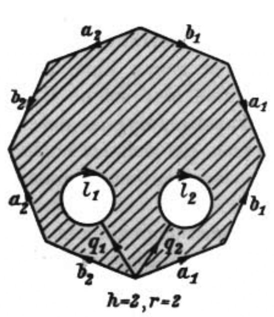


For the Klein bottle

Here is a more complicated example. We start with the octogon and after the identifications we get the double torus.



For surfaces with boundaries, the method works the same except that we allow some holes in the polygon:



Here the circles l_1 and l_2 are not identified with anything.

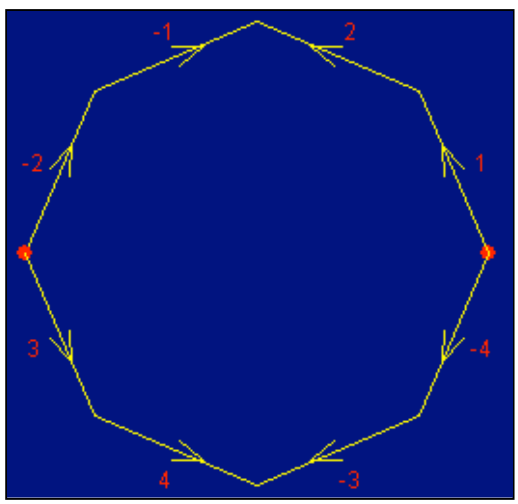
How does the applet work

- The surface should be given in the format: 1, 2, –1, 2, ... If one side is entered more than two times, the applet will not work even though it might be a closed surface.
- The applet will be working only if a correct closed (without any boundary) surface is entered. This is valid only if all of the sides entered are pairwise identified. E.g. if you enter '1' as a side of the polygon, you must enter once again (exactly once) '1' or '–1'.
- In the result **S** stands for $\mathbf{S}^2$, **P** stands for $\mathbf{RP}^2$ and **T** stands for $\mathbf{T}^2$.
- Checking *Show* will allow the step-by-step visualization of the calculation.
- The blue labels are the vertices and one can see them only if *Show* is checked.
- In the final drawing the yellow passages are tori and the blue—projective planes.

The algorithm for identifying the surface has seven steps.

Step 1

This is the initial step of the algorithm. The main purpose is to present the surface in the way 1, 2, –1, –2 etc. The different numbers correspond to different cuts in the surface and the same (or opposite) numbers correspond to identified sides of the polygon combining the directions nicely – i.e., the arrows must be in the same direction when identifying two sides.



On the picture is shown a double torus that corresponds to sequence 1, 2, –1, –2, 3, 4, –3, and –4.

Step 2

This step is again called often. It replaces all pairs of equal or opposite sequences with a pair of sides in the corresponding direction.

For instance, if we have ...1, 2, –3, 4, –4, 3, –2, –1, ... the result will be ...1, –1,...

Step 3

This step also is called often. It simplifies the polygon by removing all appearances of type *X*, *–X,* where *X* is an arbitrary integer. It is clear that when we remove such a pair, the surface will remain the same.

At this step the algorithm can finish. This will happen if the polygon consists of only two sides. Then, if they are opposite, the surface is a sphere.

Step 4

This is the most complicated step in the process. The task is to cut-and-paste the polygon in order that there remains only one vertex. So the first thing to do is to label the vertices in some manner, count them, and if there are more than one of them, perform the action.

The exact cut-and-pasting is rather complicate to explain in all detail but, for instance, it will take the sequence

1,..., 1, 2, ..., 2, ... to 1, ..., 3, ..., –1, 3, ...

This is actually cutting from the beginning of 1 to the end of 2, labeling the new side 3, and sticking the two parts along 2. This will increase by 1 the vertices labeled equally with this at the beginning of 1 and decrease these at the end of 1.

Note that the applet would rather label the new side 2 than leave it 3. This saves time to check that the number 3 is free (there is no other side labeled 3 or –3) and moreover, keeps the numbers of the sides small which means that the picture is better looking.

Step 5

On this step the twisted pairs are collected together. Once they are together they form a projective plane. This is done again by cut-and-pasting. A configuration looking like

$$1, ...(X)..., 1, ...(Y)...$$

is replaced by

$$2, 2, ...(Y)..., ...(-X)...$$

where X and Y are sequences. The actual cut is from the end of 1 to the end of the other 1. The program will again ignore the numbers and will label the new side 1.

Step 6

This step is rather similar to the previous one, with the only difference that it collects together opposing pairs. The steps till now guarantee that this can be done and the collected sides will form a torus. This time we look for

$$1, ...(X)..., 2, ...(Y)..., -1, ...(Z)..., -2, ...(T)...$$

and replace it by

$$...(Z)..., ...(Y)..., 1, 2, -1, -2, ...(X)..., ...(T)....$$

Step 7

Here everything is put together. The only essential transformation is replacing each torus by two projective planes if needed.

Some of the labels of the sides are changed for better understanding of the final result.

The applet can be found at

`http://mystic.math.neu.edu/inikolov/Surfaces/Surfaces.html`