MILNOR FIBRATIONS OF HYPERPLANE ARRANGEMENTS

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Let $f \in \mathbb{C}[z_0, \ldots, z_d]$ be a homogeneous polynomial of degree $n$.

Let $V(f) = \{z \in \mathbb{C}^{d+1} \mid f(z) = 0\}$ and $M = \mathbb{C}^{d+1} \setminus V(f)$.

The map $f: \mathbb{C}^{d+1} \to \mathbb{C}$ restricts to a map $f: M \to \mathbb{C}^*$.

This is the projection of a smooth, locally trivial bundle, known as the (global) Milnor fibration of $f$.

The typical fiber, $F = f^{-1}(1)$, is homotopic to a finite CW-complex of dim $d$. If $f$ is not a proper power, then $F$ is connected.

The monodromy of the fibration: $h: F \to F, \ z \mapsto e^{2\pi i/n} z$.

The algebraic monodromy: $h_q: H_q(F, \mathbb{C}) \to H_q(F, \mathbb{C})$. 
If \( f \) has an isolated critical point at \( 0 \), then \( F \simeq \sqrt{\mu} S^d \), where \( \mu = (n - 1)^{d+1} \).

For instance, let \( f = z_0^3 - z_1^3 \). Then \( F \) is a thrice-punctured torus (with \( h \) rotation by \( 120^\circ \)), and \( F \simeq \sqrt{4} S^1 \):

More generally, if \( f = z_0^n - z_1^n \), then \( F \) is a Riemann surface of genus \( \binom{n-1}{2} \) with \( n \) punctures, and so \( F \simeq \sqrt{(n-1)^2} S^1 \).

If the singularity at \( 0 \) is non-isolated, though, the Betti numbers \( b_q(F) \) and the algebraic monodromies \( h_q \) are hard to compute.
Let $X$ be a connected, finite cell complex, and let $\pi = \pi_1(X, x_0)$.

Let $k$ be an algebraically closed field, and let $\text{Hom}(\pi, k^*) = H^1(X, k^*)$ be the character group of $\pi$.

The (degree 1) characteristic varieties of $X$ are the jump loci for homology with coefficients in rank-1 local systems on $X$:

$$\mathcal{V}_s(X, k) = \{ \rho \in \text{Hom}(\pi, k^*) | \dim_k H_1(X, k\rho) \geq s \}. $$
Characteristic varieties

**Example (Circle)**

We have $\tilde{S}^1 = \mathbb{R}$. Identify $\pi_1(S^1, \ast) = \mathbb{Z} = \langle t \rangle$ and $k\mathbb{Z} = k[t^{\pm 1}]$. Then:

$$ C_* (\tilde{S}^1, k) : 0 \longrightarrow k[t^{\pm 1}] \xrightarrow{t^{-1}} k[t^{\pm 1}] \longrightarrow 0. $$

For $\rho \in \text{Hom}(\mathbb{Z}, k^*) = k^*$, we get

$$ C_* (\tilde{S}^1, k) \otimes_{k\mathbb{Z}} k_{\rho} : 0 \longrightarrow k \xrightarrow{\rho^{-1}} k \longrightarrow 0, $$

which is exact, except for $\rho = 1$, when $H_0(S^1, k) = H_1(S^1, k) = k$. Hence: $V_1(S^1, k) = \{1\}$ and $V_s(S^1, k) = \emptyset$, otherwise.

**Example (Punctured complex line)**

Identify $\pi_1(\mathbb{C}\setminus\{n \text{ points}\}) = F_n$, and $\text{Hom}(F_n, k^*) = (k^*)^n$. Then:

$$ V_s(\mathbb{C}\setminus\{n \text{ points}\}, k) = \begin{cases} (k^*)^n & \text{if } s < n, \\ \{1\} & \text{if } s = n, \\ \emptyset & \text{if } s > n. \end{cases} $$
Let $\pi: \mathbb{C}^{d+1}\setminus\{0\} \to \mathbb{C}\mathbb{P}^d$ be the projection map, with fiber $\mathbb{C}^*$. 

This map restricts to $\pi: M \to U$, where $U = M/\mathbb{C}^* = \mathbb{C}\mathbb{P}^d \setminus V(f)$. 

This map further restricts to a regular, $\mathbb{Z}_n$-cover $F \to U$. 

Assume $f$ is square-free, and write $f = f_1 \cdots f_r$, with factors irreducible and distinct. 

Then the cover $F \to U$ is classified by the homomorphism $\delta: \pi_1(U) \to \mathbb{Z}_n$ that sends each meridian about $V(f_i)$ to $\deg(f_i)$. 

Fix a field $k$, and let $\hat{\delta}: \text{Hom}(\mathbb{Z}_n, k^*) \to \text{Hom}(\pi_1(U), k^*)$ be the induced homomorphism between character groups. 

If $\text{char}(k) \nmid n$, then

$$\dim_k H_1(F, k) = \sum_{s \geq 1} \left| V_s(U, k) \cap \text{im}(\hat{\delta}) \right|.$$
HYPERPLANE ARRANGEMENTS

- $\mathcal{A}$: A (central) arrangement of hyperplanes in $\mathbb{C}^{d+1}$.

Intersection lattice: $L(\mathcal{A})$.

Complement: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$.

The Boolean arrangement $\mathcal{B}_n$
  - $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
  - $L(\mathcal{B}_n)$: lattice of subsets of $\{0, 1\}^n$.
  - $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n$.

The braid arrangement $\mathcal{A}_n$ (or, reflection arr. of type $A_{n-1}$)
  - $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
  - $L(\mathcal{A}_n)$: lattice of partitions of $[n] = \{1, \ldots, n\}$.
  - $M(\mathcal{A}_n)$: configuration space of $n$ ordered points in $\mathbb{C}$ (a classifying space for the pure braid group on $n$ strings).
\( M \) has the homotopy type of a connected, finite CW-complex of dimension \( d + 1 \). In fact, \( M \) admits a minimal cell structure.

In particular, \( H_*(M, \mathbb{Z}) \) is torsion-free. The Betti numbers \( b_q(M) := \text{rank } H_q(M, \mathbb{Z}) \) are given by the Möbius function of \( L(\mathcal{A}) \).

The Orlik–Solomon algebra \( A = H^*(M, \mathbb{Z}) \) is determined by \( L(\mathcal{A}) \), but \( \pi_1(M) \) is not.
For each $H \in \mathcal{A}$, let $f_H : \mathbb{C}^{d+1} \rightarrow \mathbb{C}$ be a linear form with kernel $H$.

Let $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} f_H$, a homogeneous polynomial of degree $n$.

This polynomial defines the Milnor fibration of $\mathcal{A}$, with fiber $F = F(\mathcal{A})$.

**Example**

Let $\mathcal{B}_n$ be the Boolean arrangement, with $Q = z_1 \cdots z_n$. Then $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$ and $F(\mathcal{B}_n) = \ker(Q) \cong (\mathbb{C}^*)^{n-1}$.

Let $\mathcal{A}$ be an arrangement of planes in $\mathbb{C}^3$. Its projectivization, $\bar{\mathcal{A}}$, is an arrangement of lines in $\mathbb{CP}^2$.

A flat $X \in L_2(\mathcal{A})$ has multiplicity $q$ if $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \supset H\}$ has size $q$, i.e., there are exactly $q$ lines from $\bar{\mathcal{A}}$ passing through $\bar{X}$. 
Question: Are the Betti numbers of $F(A)$ and the characteristic polynomial of the algebraic monodromy determined by $L(A)$? Let $\Delta_A(t) := \det(h_1 - t \cdot \text{id})$. Then $b_1(F(A)) = \deg \Delta_A$.

**Theorem (Papadima–S. 2013)**

Suppose all flats $X \in L_2(A)$ have multiplicity 2 or 3. Then $\Delta_A(t)$, and thus $b_1(F(A))$, are combinatorially determined.

- We relate the cohomology jump loci of $M(A)$ in characteristic $p$ with those in characteristic 0.

- The bridge between the two goes through the representation variety $\text{Hom}_{\text{Lie}}(h(A), sl_2)$.

- A key combinatorial ingredient is the notion of multinet.
Let $A = H^*(M(A), \mathbb{k})$ — an algebra that depends only on $L(A)$ (and the field $\mathbb{k}$).

For each $a \in A^1$, we have $a^2 = 0$. Thus, we get a cochain complex, $(A, \cdot a): A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{a} \cdots$

The (degree 1) resonance varieties of $\mathcal{A}$ are the cohomology jump loci of this “Aomoto complex”:

$$\mathcal{R}_s(\mathcal{A}, \mathbb{k}) = \{ a \in A^1 \mid \dim_{\mathbb{k}} H^1(A, \cdot a) \geq s \},$$

In particular, $a \in A^1$ belongs to $\mathcal{R}_1(\mathcal{A}, \mathbb{k})$ iff there is $b \in A^1$ not proportional to $a$, such that $a \cup b = 0$ in $A^2$. 

Now assume $\mathbb{k}$ has characteristic $p > 0$.

Let $\sigma = \sum_{H \in \mathcal{A}} e_H \in A^1$ be the “diagonal” vector, and define

$$\beta_p(\mathcal{A}) = \dim_{\mathbb{k}} H^1(\mathcal{A}, \cdot \sigma).$$

That is, $\beta_p(\mathcal{A}) = \max\{s \mid \sigma \in R^1_s(\mathcal{A}, \mathbb{k})\}$.

Clearly, $\beta_p(\mathcal{A})$ depends only on $L(\mathcal{A})$ and $p$. Moreover, $0 \leq \beta_p(\mathcal{A}) \leq |\mathcal{A}| - 2$.

**Theorem**

*If $L_2(\mathcal{A})$ has no flats of multiplicity $3r$ with $r > 1$, then $\beta_3(\mathcal{A}) \leq 2$.***

For each $m \geq 1$, there is a matroid $\mathcal{M}_m$ with all rank 2 flats of multiplicity 3, and such that $\beta_3(\mathcal{M}_m) = m$.

- $\mathcal{M}_1$: pencil of 3 lines. $\mathcal{M}_2$: Ceva arrangement.
- $\mathcal{M}_m$ with $m > 2$: not realizable over $\mathbb{C}$.  

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The monodromy $h: F(\mathcal{A}) \to F(\mathcal{A})$ has order $n = |\mathcal{A}|$. Thus,

$$\Delta_{\mathcal{A}}(t) = \prod_{d|n} \Phi_d(t)^{e_d(\mathcal{A})},$$

where $\Phi_1 = t - 1$, $\Phi_2 = t + 1$, $\Phi_3 = t^2 + t + 1$, $\Phi_4 = t^2 + 1$, ... are the cyclotomic polynomials, and $e_d(\mathcal{A}) \in \mathbb{Z}_{\geq 0}$.

Easy to see: $e_1(\mathcal{A}) = n - 1$. Hence, $H_1(F(\mathcal{A}), \mathbb{C})$, when viewed as a module over $\mathbb{C}[\mathbb{Z}_n]$, decomposes as

$$(\mathbb{C}[t] / (t - 1))^{n-1} \oplus \bigoplus_{1 < d|n} (\mathbb{C}[t] / \Phi_d(t))^{e_d(\mathcal{A})}.$$

In particular, $b_1(F(\mathcal{A})) = n - 1 + \sum_{1 < d|n} \varphi(d) e_d(\mathcal{A})$. 
Thus, in degree 1, question (Q1) is equivalent to: are the integers $e_d(A)$ determined by $L_{\leq 2}(A)$?

Not all divisors of $n$ appear in the above formulas: If $d$ does not divide $|A_X|$, for some $X \in L_2(A)$, then $e_d(A) = 0$ (Libgober).

In particular, if $L_2(A)$ has only flats of multiplicity 2 and 3, then $\Delta_A(t) = (t - 1)^{n-1}(t^2 + t + 1)^{e_3}$.

If multiplicity 4 appears, then also get factor of $(t + 1)^{e_2} \cdot (t^2 + 1)^{e_4}$.


$e_{ps}(A) \leq \beta_p(A)$, for all $s \geq 1$. 

Theorem

Suppose $L_2(\mathcal{A})$ has no flats of multiplicity $3r$, with $r > 1$. Then $e_3(\mathcal{A}) = \beta_3(\mathcal{A})$, and thus $e_3(\mathcal{A})$ is combinatorially determined.

A similar result holds for $e_2(\mathcal{A})$ and $e_4(\mathcal{A})$, under some additional hypothesis.

Corollary

If $\mathcal{A}$ is an arrangement of $n$ lines in $\mathbb{P}^2$ with only double and triple points, then $\Delta_{\mathcal{A}}(t) = (t - 1)^{n-1}(t^2 + t + 1)^{\beta_3(\mathcal{A})}$ is combinatorially determined.

Corollary (Libgober 2012)

If $\mathcal{A}$ is an arrangement of $n$ lines in $\mathbb{P}^2$ with only double and triple points, then the question whether $\Delta_{\mathcal{A}}(t) = (t - 1)^{n-1}$ or not is combinatorially determined.
**Conjecture**

Let $\mathcal{A}$ be an essential arrangement in $\mathbb{C}^3$. Then

$$\Delta_{\mathcal{A}}(t) = (t - 1)^{|\mathcal{A}| - 1}(t^2 + t + 1)\beta_3(\mathcal{A})[(t + 1)(t^2 + 1)]\beta_2(\mathcal{A}),$$

where $\beta_3(\mathcal{A}) \in \{0, 1, 2\}$ and $\beta_2(\mathcal{A}) \in \{0, 2\}$

Compare this conjecture with

**Conjecture (Yoshinaga 2013)**

Assume $\mathcal{A}$ is a simplicial arrangement. Then

$$\Delta_{\mathcal{A}}(t) = (t - 1)^{|\mathcal{A}| - 1}(t^2 + t + 1)e_3(\mathcal{A}),$$

where $e_3(\mathcal{A}) \in \{0, 1\}$. 
**Definition (Falk and Yuzvinsky)**

A *multinet* on $\mathcal{A}$ is a partition of the set $\mathcal{A}$ into $k \geq 3$ subsets $\mathcal{A}_1, \ldots, \mathcal{A}_k$, together with an assignment of multiplicities, $m: \mathcal{A} \to \mathbb{N}$, and a subset $\mathcal{X} \subseteq L_2(\mathcal{A})$, called the base locus, such that:

1. There is an integer $d$ such that $\sum_{H \in \mathcal{A}_\alpha} m_H = d$, for all $\alpha \in [k]$.
2. If $H$ and $H'$ are in different classes, then $H \cap H' \in \mathcal{X}$.
3. For each $X \in \mathcal{X}$, the sum $n_X = \sum_{H \in \mathcal{A}_\alpha: H \supseteq X} m_H$ is independent of $\alpha$.
4. Each set $(\bigcup_{H \in \mathcal{A}_\alpha} H) \setminus \mathcal{X}$ is connected.

- A similar definition can be made for any (rank 3) matroid.
- A multinet as above is also called a $(k, d)$-multinet, or a $k$-multinet.
- The multinet is *reduced* if $m_H = 1$, for all $H \in \mathcal{A}$. 
A *net* is a reduced multinet with $n_X = 1$, for all $X \in \mathcal{X}$.

In this case, $|A_\alpha| = |A| / k = d$, for all $\alpha$.

Moreover, $\mathring{\mathcal{X}}$ has size $d^2$, and is encoded by a $(k - 2)$-tuple of orthogonal Latin squares.

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A (3, 2)-net on the $A_3$ arrangement $\mathring{\mathcal{X}}$ consists of 4 triple points ($n_X = 1$) and 3 triple points ($n_X = 2$). A (3, 4)-multinet on the $B_3$ arrangement $\mathring{\mathcal{X}}$ consists of 4 triple points ($n_X = 1$).
A \((3, 3)\)-net on the Ceva matroid. A \((4, 3)\)-net on the Hessian matroid.
If \( \mathcal{A} \) has no flats of multiplicity \( kr \), for some \( r > 1 \), then every reduced \( k \)-multinet is a \( k \)-net.

(Kawahara): given any Latin square, there is a matroid \( \mathcal{M} \) with a 3-net \((\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)\) realizing it, such that each \( \mathcal{M}_\alpha \) is uniform.

(Yuzvinsky and Pereira–Yuz): If \( \mathcal{A} \) supports a \( k \)-multinet with \(|\mathcal{X}| > 1\), then \( k = 3 \) or 4; if the multinet is not reduced, then \( k = 3 \).

(Wakefield & al): The only \((4, 3)\)-net in \( \mathbb{C}P^2 \) is the Hessian; there are no \((4, 4)\), \((4, 5)\), or \((4, 6)\) nets in \( \mathbb{C}P^2 \).

Conjecture (Yuz): The only 4-multinet is the Hessian \((4, 3)\)-net.
**Lemma**

If \( \mathcal{A} \) supports a 3-net with parts \( \mathcal{A}_\alpha \), then:

1. \( 1 \leq \beta_3(\mathcal{A}) \leq \beta_3(\mathcal{A}_\alpha) + 1 \), for all \( \alpha \).
2. If \( \beta_3(\mathcal{A}_\alpha) = 0 \), for some \( \alpha \), then \( \beta_3(\mathcal{A}) = 1 \).
3. If \( \beta_3(\mathcal{A}_\alpha) = 1 \), for some \( \alpha \), then \( \beta_3(\mathcal{A}) = 1 \) or 2.

All possibilities do occur:

- Braid arrangement: has a \( (3, 2) \)-net from the Latin square of \( \mathbb{Z}_2 \).
  \[ \beta_3(\mathcal{A}_\alpha) = 0 \quad (\forall \alpha) \quad \text{and} \quad \beta_3(\mathcal{A}) = 1. \]

- Pappus arrangement: has a \( (3, 3) \)-net from the Latin square of \( \mathbb{Z}_3 \).
  \[ \beta_3(\mathcal{A}_1) = \beta_3(\mathcal{A}_2) = 0, \quad \beta_3(\mathcal{A}_3) = 1 \quad \text{and} \quad \beta_3(\mathcal{A}) = 1. \]

- Ceva arrangement: has a \( (3, 3) \)-net from the Latin square of \( \mathbb{Z}_3 \).
  \[ \beta_3(\mathcal{A}_\alpha) = 1 \quad (\forall \alpha) \quad \text{and} \quad \beta_3(\mathcal{A}) = 2. \]
Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^3$. Work of Arapura, Falk, Cohen–S., Libgober–Yuz, Falk–Yuz completely describes the varieties $\mathcal{R}_s(\mathcal{A}, \mathbb{C})$: 

- $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$ is a union of linear subspaces in $H^1(M(\mathcal{A}), \mathbb{C}) = \mathbb{C}^{\mid \mathcal{A} \mid}$.

- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.

- $\mathcal{R}_s(\mathcal{A}, \mathbb{C})$ is the union of those linear subspaces that have dimension at least $s + 1$. 

• Each flat $X \in L_2(A)$ of multiplicity $k \geq 3$ gives rise to a local component of $R_1(A, C)$, of dimension $k - 1$.

• More generally, every $k$-multinet on a sub-arrangement $B \subseteq A$ gives rise to a component of dimension $k - 1$, and all components of $R_1(A, C)$ arise in this way.

• Note: the varieties $R_1(A, k)$ with $\text{char}(k) > 0$ can be more complicated: components may be non-linear, and they may intersect non-transversely.

**Theorem**

Suppose $L_2(A)$ has no flats of multiplicity $3r$, with $r > 1$. Then $R_1(A, C)$ has at least $(3^{β_3(A)} - 1)/2$ essential components, all corresponding to $3$-nets.
Work of Arapura, Libgober, Cohen–S., S., Libgober–Yuz, Falk–Yuz, Dimca, Dimca–Papadima–S., Artal–Cogolludo–Matei, Budur–Wang ... provides a fairly explicit description of the varieties $\mathcal{V}_S(\mathcal{A}, \mathbb{C})$:

- Each variety $\mathcal{V}_S(\mathcal{A}, \mathbb{C})$ is a finite union of torsion-translates of algebraic subtori of $(\mathbb{C}^*)^n$.
- If a linear subspace $L \subset \mathbb{C}^n$ is a component of $\mathcal{R}_S(\mathcal{A}, \mathbb{C})$, then the algebraic torus $T = \exp(L)$ is a component of $\mathcal{V}_S(\mathcal{A}, \mathbb{C})$.
- Moreover, $T = f^*(H^1(S, \mathbb{C}^*))$, where $f: M(\mathcal{A}) \to S$ is an orbifold fibration, with base $S = \mathbb{C}P^1 \setminus \{k \text{ points}\}$, for some $k \geq 3$.
- All components of $\mathcal{V}_S(\mathcal{A}, \mathbb{C})$ passing through the origin $1 \in (\mathbb{C}^*)^n$ arise in this way (and thus, are combinatorially determined).

**Theorem**

*If* $\mathcal{A}$ *admits a reduced* $k$-*multinet, then* $e_k(\mathcal{A}) \geq k - 2$. 

**ALEX SUCIU (NORTHEASTERN)**

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Main theorem

Theorem

Suppose $L_2(A)$ has no flats of multiplicity $3r$ with $r > 1$. Then TFAE:

1. $L_{\leq 2}(A)$ admits a reduced $3$-multinet.
2. $L_{\leq 2}(A)$ admits a $3$-net.
3. $\beta_3(A) \neq 0$.
4. $e_3(A) \neq 0$.

Moreover, $\beta_3(A) \leq 2$ and $\beta_3(A) = e_3(A)$.

- $(2) \Rightarrow (1)$: obvious.
- $(1) \Rightarrow (4)$: by above theorem.
- $(4) \Rightarrow (3)$: by modular bound $e_p(A) \leq \beta_p(A)$.
- $(3) \Rightarrow (2)$: use flat, $\mathfrak{sl}_2$-valued connections on the OS-algebra.
- $\beta_3(A) \leq 2$: a previous theorem.
- Last assertion: put things together.
Some ingredients in the proof:

- Let $A$ be a graded, graded-commutative algebra over $\mathbb{C}$. Assume $\dim A^i < \infty$ and $A^0 = \mathbb{C}$.

- Let $g$ be a finite-dimensional Lie algebra over $\mathbb{C}$. On $A \otimes g$, set $[a \otimes x, b \otimes y] = ab \otimes [x, y]$.

- Define the space of flat, $g$-valued connections on $A$ as
  \[ \mathcal{F}(A, g) = \{ \omega \in A^1 \otimes g \mid [\omega, \omega] = 0 \}. \]

- Alternatively, define the holonomy Lie algebra of $A$ as
  \[ \mathfrak{h}(A) = \text{Lie}(A_1) / (\text{im}(\nabla)). \]
  where $\nabla : A_2 \to A_1 \wedge A_1$ is the dual to the multiplication map.

- Then, the canonical isomorphism $A^1 \otimes g \cong \text{Hom}_{\mathbb{C}}(A_1, g)$ restricts to a functorial isomorphism
  \[ \mathcal{F}(A, g) \cong \text{Hom}_{\text{Lie}}(\mathfrak{h}(A), g). \]
Given a linear subspace $P \subset A^1$, define a sub-algebra $A_P \subset A^{\leq 2}$ by setting $A^1_P = P$, $A^2_P = A^2$ and restricting the multiplication map.

**Theorem (Macinic, Papadadima, Popescu, S. 2013)**

Suppose $\mathcal{R}_1(A) = \bigcup_{P \in \mathcal{P}} P$, where $\mathcal{P}$ is a finite collection of linear subspaces of $A^1$, intersecting pairwise only at $0$. Then:

1. $\mathcal{F}(A_P, g) \cap \mathcal{F}(A_{P'}, g) = \{0\}$, for all distinct subspaces $P, P' \in \mathcal{P}$.
2. $\mathcal{F}(A, g) \supseteq \mathcal{F}^{(1)}(A, g) \cup \bigcup_{P \in \mathcal{P}} \mathcal{F}(A_P, g)$.
3. If $g = sl_2$, then the above inclusion holds as an equality.

- Given a vector space $V$, and a finite set $I$, let
  $$\mathcal{H}^I(V) = \{ x = (x_i) \in V^I \mid \sum_{i \in I} x_i = 0 \}.$$ 

- View each $x \in V^I$ as a map $x : I \to V$. For a fixed $\tau \in I^A$, we obtain a linear "evaluation" map,
  $$\text{ev}_\tau : V^I \to V^A, \quad \text{ev}_\tau(x)_u = x_{\tau(u)}, \text{ for } u \in A.$$
Suppose $L_2(A)$ does not have flats of multiplicity $3r$, for any $r > 1$. Suppose $\beta_3(A) \neq 0$, i.e., there is $\tau \in H^1(M(A), \mathbb{F}_3)$ non-constant, such that $\tau \cup \sigma = 0$. Then:

1. The evaluation map $\text{ev}_\tau : g^{F_3} \to g^A$ defines an algebraic map

$$\text{ev}_\tau : \mathcal{H}^{F_3}(g) \to \text{Hom}_{\text{Lie}}(h(A), g),$$

taking regular elements to regular elements.

2. There is an integer $k \geq 3$ and a $k$-multinet $\mathcal{N} = \mathcal{N}(\tau)$ on $A$, unique up to the natural $\Sigma_k$-action, with associated admissible map $f_\mathcal{N} : M(A) \to S = \mathbb{CP}^1 \setminus \{k \text{ points}\}$, such that $\text{ev}_\tau(\mathcal{H}^{F_3}(s l_2))$ is contained in the image of

$$(f_\mathcal{N}^*)^! : \text{Hom}_{\text{Lie}}(h(S), s l_2) \to \text{Hom}_{\text{Lie}}(h(A), s l_2).$$

With some more work, it can be shown that this $3$-multinet is a $3$-net.