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Mini-Workshop on
Interactions between low-dimensional topology and complex algebraic geometry

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A recurring theme in topology is to determine the geometric and homological finiteness properties of spaces and groups.

For instance, to decide whether a path-connected space $X$ is homotopy equivalent to a CW-complex with finite $k$-skeleton.

A group $G$ has property $F_k$ if it admits a classifying space $K(G, 1)$ with finite $k$-skeleton.

- $F_1$: $G$ is finitely generated;
- $F_2$: $G$ is finitely presentable.

$G$ has property $FP_k$ if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admits a projective $\mathbb{Z}G$-resolution which is finitely generated in all dimensions up to $k$.

The following implications (none of which can be reversed) hold:

- $G$ is of type $F_k$ $\Rightarrow$ $G$ is of type $FP_k$
  $\Rightarrow H_i(G, \mathbb{Z})$ is finitely generated, for all $i \leq k$
  $\Rightarrow b_i(G) < \infty$, for all $i \leq k$.

- Moreover, $FP_k \& F_2 \Rightarrow F_k$. 
(Bieri–Neumann–Strebel 1987) For a f.g. group $G$, let

$$\Sigma^1(G) = \{ \chi \in S(G) \mid C_\chi(G) \text{ is connected} \},$$

where $S(G) = (\text{Hom}(G, \mathbb{R}) \setminus \{0\}) / \mathbb{R}^+$ and $C_\chi(G)$ is the induced subgraph of $\text{Cay}(G)$ on vertex set $G_\chi = \{ g \in G \mid \chi(g) \geq 0 \}$.

$\Sigma^1(G)$ is an open set, independent of generating set for $G$.

(Bieri, Renz 1988)

$$\Sigma^k(G, \mathbb{Z}) = \{ \chi \in S(G) \mid \text{the monoid } G_\chi \text{ is of type } FP_k \}.$$ 

In particular, $\Sigma^1(G, \mathbb{Z}) = \Sigma^1(G)$.

The $\Sigma$-invariants control the finiteness properties of normal subgroups $N \triangleleft G$ for which $G/N$ is free abelian:

$$N \text{ is of type } FP_k \iff S(G, N) \subseteq \Sigma^k(G, \mathbb{Z})$$

where $S(G, N) = \{ \chi \in S(G) \mid \chi(N) = 0 \}$. In particular:

$$\ker(\chi : G \to \mathbb{Z}) \text{ is f.g. } \iff \{ \pm \chi \} \subseteq \Sigma^1(G).$$
• Fix a connected CW-complex $X$ with finite $k$-skeleton, for some $k \geq 1$. Let $G = \pi_1(X, x_0)$.

• For each $\chi \in S(X) := S(G)$, set

$$\widehat{\mathbb{Z}G}_\chi = \left\{ \lambda \in \mathbb{Z}^G \mid \{ g \in \text{supp } \lambda \mid \chi(g) < c \} \text{ is finite}, \forall c \in \mathbb{R} \right\}.$$ 

This is a ring, contains $\mathbb{Z}G$ as a subring; hence, a $\mathbb{Z}G$-module.

• (Farber, Geoghegan, Schütz 2010)

$$\Sigma^q(X, \mathbb{Z}) := \{ \chi \in S(X) \mid H_i(X, \widehat{\mathbb{Z}G}_-\chi) = 0, \forall i \leq q \}.$$ 

• (Bieri) $G$ is of type $\text{FP}_k \implies \Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G, 1), \mathbb{Z}), \forall q \leq k.$
Dwyer–Fried sets

For a fixed $r \in \mathbb{N}$, the connected, regular covers $Y \to X$ with group of deck-transformations $\mathbb{Z}^r$ are parametrized by the Grassmannian of $r$-planes in $H^1(X, \mathbb{Q})$.

Moving about this variety, and recording when $b_1(Y), \ldots, b_i(Y)$ are finite defines subsets $\Omega^i_r(X) \subseteq \text{Gr}_r(H^1(X, \mathbb{Q}))$, which we call the Dwyer–Fried invariants of $X$.

These sets depend only on the homotopy type of $X$. Hence, if $G$ is a f.g. group, we may define $\Omega^i_r(G) := \Omega^i_r(K(G, 1))$.

Example

Let $K$ be a knot in $S^3$. If $X = S^3 \setminus K$, then $\dim_{\mathbb{Q}} H_1(X^{ab}, \mathbb{Q}) < \infty$, and so $\Omega^1_1(X) = \{\text{pt}\}$. But $H_1(X^{ab}, \mathbb{Z})$ need not be a f.g. $\mathbb{Z}$-module.
**Theorem**

Let $G$ be a f.g. group, and $\nu: G \to \mathbb{Z}^r$ an epimorphism, with kernel $\Gamma$. Suppose $\Omega^k_r(G) = \emptyset$, and $\Gamma$ is of type $F_{k-1}$. Then $b_k(\Gamma) = \infty$.

Proof: Set $X = K(G, 1)$; then $X^\nu = K(\Gamma, 1)$. Since $\Gamma$ is of type $F_{k-1}$, we have $b_i(X^\nu) < \infty$ for $i \leq k - 1$. Since $\Omega^k_r(X) = \emptyset$, we must have $b_k(X^\nu) = \infty$.

It follows that $H_k(\Gamma, \mathbb{Z})$ is not f.g., and $\Gamma$ is not of type $FP_k$.

**Corollary**

Let $G$ be a f.g. group, and suppose $\Omega^3_1(G) = \emptyset$. Let $\nu: G \to \mathbb{Z}$ be an epimorphism. If the group $\Gamma = \ker(\nu)$ is f.p., then $b_3(\Gamma) = \infty$. 
The Stallings group

- Let $Y = S^1 \lor S^1$ and $X = Y \times Y \times Y$. Clearly, $X$ is a classifying space for $G = F_2 \times F_2 \times F_2$.

- Let $\nu: G \to \mathbb{Z}$ be the homomorphism taking each standard generator to 1. Set $\Gamma = \ker(\nu)$.

- Stallings (1963) showed that $\Gamma$ is finitely presented:

$$\Gamma = \langle a, b, c, x, y \mid [x, a], [y, a], [x, b], [y, b], [a^{-1} x, c], [a^{-1} y, c], [b^{-1} a, c]\rangle$$

- Stallings then showed, via a Mayer-Vietoris argument, that $H_3(\Gamma, \mathbb{Z})$ is not finitely generated.

- Alternate explanation: $\Omega^3_1(X) = \emptyset$. Thus, by the previous Corollary, a stronger statement holds: $b_3(\Gamma)$ is not finite.
Kollár’s question

**Question (J. Kollár 1995)**

Given a smooth, projective variety $M$, is the fundamental group $G = \pi_1(M)$ commensurable, up to finite kernels, with another group, $\pi$, admitting a $K(\pi, 1)$ which is a quasi-projective variety?

(Two groups, $G_1$ and $G_2$, are said to be *commensurable up to finite kernels* if there is a zig-zag of groups and homomorphisms connecting them, with all arrows of finite kernel and cofinite image.)

**Theorem (Dimca–Papadima–S. 2009)**

For each $k \geq 3$, there is a smooth, irreducible, complex projective variety $M$ of complex dimension $k - 1$, such that $\pi_1(M)$ is of type $F_{k-1}$, but not of type $FP_k$.

Further examples given by Llosa Isenrich and Bridson (2016/17).
Let $A = (A^\bullet, d)$ be a commutative, differential graded algebra over a field $\mathbb{k}$ of characteristic 0. That is:

- $A = \bigoplus_{i \geq 0} A^i$, where $A^i$ are $\mathbb{k}$-vector spaces.
- The multiplication $\cdot : A^i \otimes A^j \to A^{i+j}$ is graded-commutative, i.e., $ab = (-1)^{|a||b|}ba$ for all homogeneous $a$ and $b$.
- The differential $d : A^i \to A^{i+1}$ satisfies the graded Leibnitz rule, i.e., $d(ab) = d(a)b + (-1)^{|a|}a d(b)$.

A CDGA $A$ is of \textit{finite-type} (or $q$-\textit{finite}) if it is connected (i.e., $A^0 = \mathbb{k} \cdot 1$) and $\dim A^i < \infty$ for all $i \leq q$.

$H^\bullet(A)$ inherits an algebra structure from $A$.

A cdga morphism $\varphi : A \to B$ is both an algebra map and a cochain map. Hence, it induces a morphism $\varphi^* : H^\bullet(A) \to H^\bullet(B)$. 
A map \( \varphi: A \to B \) is a quasi-isomorphism if \( \varphi^* \) is an isomorphism. Likewise, \( \varphi \) is a \( q \)-quasi-isomorphism (for some \( q \geq 1 \)) if \( \varphi^* \) is an isomorphism in degrees \( \leq q \) and is injective in degree \( q + 1 \).

Two cdgas, \( A \) and \( B \), are \((q-)equivalent \) \( (\simeq_q)\) if there is a zig-zag of \((q-)\)quasi-isomorphisms connecting \( A \) to \( B \).

A cdga \( A \) is formal (or just \( q \)-formal) if it is \((q-)\)equivalent to \((H^\bullet(A), d = 0)\).

A cdga is \( q \)-minimal if it is of the form \((\bigwedge V, d)\), where the differential structure is the inductive limit of a sequence of Hirsch extensions of increasing degrees, and \( V^i = 0 \) for \( i > q \).

Every cdga \( A \) with \( H^0(A) = \mathbb{k} \) admits a \( q \)-minimal model, \( \mathcal{M}_q(A) \) (i.e., a \( q \)-equivalence \( \mathcal{M}_q(A) \to A \) with \( \mathcal{M}_q(A) = (\bigwedge V, d) \) a \( q \)-minimal cdga), unique up to iso.
Given any (path-connected) space $X$, there is an associated Sullivan $\mathbb{Q}$-cdga, $A_{\text{PL}}(X)$, such that $H^\bullet(A_{\text{PL}}(X)) = H^\bullet(X, \mathbb{Q})$.

An algebraic ($q$-)model (over $k$) for $X$ is a $k$-cdga $(A, d)$ which is ($q$-) equivalent to $A_{\text{PL}}(X) \otimes_{\mathbb{Q}} k$.

If $M$ is a smooth manifold, then $\Omega_{dR}(M)$ is a model for $M$ (over $\mathbb{R}$).

Examples of spaces having finite-type models include:

- Formal spaces (such as compact Kähler manifolds, hyperplane arrangement complements, toric spaces, etc).
- Smooth quasi-projective varieties, compact solvmanifolds, Sasakian manifolds, etc.
Let \( \hat{G} = \text{Hom}(G, \mathbb{C}^*) = H^1(X, \mathbb{C}^*) \) be the character group of \( G = \pi_1(X) \).

The characteristic varieties of \( X \) are the sets

\[
\mathcal{V}^i(X) = \{ \rho \in \hat{G} \mid H_i(X, \mathbb{C}_\rho) \neq 0 \}.
\]

If \( X \) has finite \( k \)-skeleton, then \( \mathcal{V}^i(X) \) is a Zariski closed subset of the algebraic group \( \hat{G} \), for each \( i \leq k \).

The varieties \( \mathcal{V}^i(X) \) are homotopy-type invariants of \( X \).

\( \mathcal{V}^1(X) \) depends only on \( G = \pi_1(X) \). Set \( \mathcal{V}^i(G) := \mathcal{V}^i(K(G,1)) \). Then \( \mathcal{V}^1(G) = \mathcal{V}^1(G/G'') \).

**Example (S.–Yang–Zhang – 2015)**

Let \( f \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \) be an Laurent polynomial with \( f(1) = 0 \). There is then a f.p. group \( G \) with \( G_{ab} = \mathbb{Z}^n \) such that \( \mathcal{V}^1(G) = \mathcal{V}(f) \).
Let $A = (A^\bullet, d)$ be a connected, finite-type CDGA over $\mathbb{C}$.

For each $a \in Z^1(A) \cong H^1(A)$, we get a cochain complex,

$$(A^\bullet, \delta_a): A^0 \overset{\delta_a^0}{\to} A^1 \overset{\delta_a^1}{\to} A^2 \overset{\delta_a^2}{\to} \cdots,$$

with differentials $\delta_a^i(u) = a \cdot u + d u$, for all $u \in A^i$.

The resonance varieties of $A$ are the affine varieties

$$\mathcal{R}^i(A) = \{ a \in H^1(A) \mid H^i(A^\bullet, \delta_a) \neq 0 \}.$$

If $X$ is a connected, finite-type CW-complex, we get the usual resonance varieties by setting $\mathcal{R}^i(X) := \mathcal{R}^i(H^\bullet(X, \mathbb{C}))$. 
**Question**

Let $X$ be a connected CW-complex with finite $q$-skeleton. Does $X$ admit a $q$-finite $q$-model $A$?

**Theorem**

If $X$ is as above, then, for all $i \leq q$:

- (Dimca–Papadima 2014) $\mathcal{V}^i(X)_{(1)} \cong \mathcal{R}^i(A)_{(0)}$.
  
  In particular, if $X$ is $q$-formal, then $\mathcal{V}^i(X)_{(1)} \cong \mathcal{R}^i(X)_{(0)}$.

- (Macinic, Papadima, Popescu, S. 2017) $TC_0(\mathcal{R}^i(A)) \subseteq \mathcal{R}^i(X)$.

- (Budur–Wang 2017) All the irreducible components of $\mathcal{V}^i(X)$ passing through the origin of $H^1(X, \mathbb{C}^*)$ are algebraic subtori.

**Example**

Let $G$ be a f.p. group with $G_{ab} = \mathbb{Z}^n$ and $\mathcal{V}^1(G) = \{ t \in (\mathbb{C}^*)^n \mid \sum_{i=1}^{n} t_i = n \}$. Then $G$ admits no 1-finite 1-model.
**Theorem (Papadima–S. 2017)**

Suppose $X$ is $(q + 1)$ finite, or $X$ admits a $q$-finite $q$-model. Then $b_i(\mathcal{M}_q(X)) < \infty$, for all $i \leq q + 1$.

**Corollary**

Let $G$ be a f.g. group. Assume that either $G$ is finitely presented, or $G$ has a $1$-finite $1$-model. Then $b_2(\mathcal{M}_1(G)) < \infty$.

**Example**

- Consider the free metabelian group $G = F_n / F''_n$ with $n \geq 2$.
- We have $\mathcal{V}^1(G) = \mathcal{V}^1(F_n) = (\mathbb{C}^*)^n$, and so $G$ passes the Budur–Wang test.
- But $b_2(\mathcal{M}_1(G)) = \infty$, and so $G$ admits no $1$-finite $1$-model (and is not finitely presented).
Bounding the $\Sigma$-invariants

- Let $\exp : H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^*)$ be the coefficient homomorphism induced by $\mathbb{C} \to \mathbb{C}^*, z \mapsto e^z$.
- Given a Zariski closed subset $W \subset H^1(X, \mathbb{C}^*)$, set
  \[ \tau_1(W) = \{ z \in H^1(X, \mathbb{C}) \mid \exp(\lambda z) \in W, \forall \lambda \in \mathbb{C} \}. \]
- $\tau_1(W)$ is a finite union of rationally defined linear subspaces.
- Set $\tau_1^k(W) = \tau_1(W) \cap H^1(X, \mathbb{C}^*)$ for $k \in \mathbb{C}$; $\mathcal{V}i(X) = \bigcup_{j \leq i} \mathcal{V}j(X)$.

**Theorem (Papadima–S. 2010)**

\[ \Sigma^i(X, \mathbb{Z}) \subseteq S(X) \setminus S(\tau_1^R(\mathcal{V}i(X))). \] (†)

If $X$ is formal, we may replace $\tau_1^R(\mathcal{V}i(X))$ with $\bigcup_{j \leq i} \mathcal{R}j(X, \mathbb{R})$.

**Example (Koban–McCammond–Meier 2015)**

\[ \Sigma^1(P_n) = \mathcal{R}1(P_n, \mathbb{R})^\mathbb{C}. \]
Bounding the $\Omega$-invariants

**Theorem (Dwyer–Fried 1987, Papadima–S. 2010)**

Let $\nu: \pi_1(X) \to \mathbb{Z}^r$ be an epimorphism. Then $\bigoplus_{i=0}^{k} H_i(X^\nu, \mathbb{C})$ is finite-dimensional if and only if the algebraic torus $\text{im} \left( \hat{\nu}: \hat{\mathbb{Z}}^r \to \hat{\pi}_1(X) \right)$ intersects $\mathcal{W}^k(X)$ in only finitely many points.

**Corollary (S. 2014)**

$$\Omega^i_r(X) = \{ P \in \text{Gr}_r(H^1(X, \mathbb{Q})) \mid \dim (\exp(P \otimes \mathbb{C}) \cap \mathcal{W}^i(X)) = 0 \}.$$  

Given a homogeneous variety $V \subset \mathbb{k}^n$, the set $\sigma_r(V) = \{ P \in \text{Gr}_r(\mathbb{k}^n) \mid P \cap V \neq \{0\} \}$ is Zariski closed.

**Theorem (S. 2012/2014)**

$$\Omega^i_r(X) \subseteq \text{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r(\tau^Q_1(\mathcal{W}^i(X))).$$

If the upper bound for the $\Sigma$-invariants is attained, then the upper bound for the $\Omega$-invariants is also attained.
Let $G$ be a f.g. group and let $p$ be a prime.

We say that $G$ is *residually finite rationally $p$* if there exists a sequence of subgroups $G = G_0 > \cdots > G_i > G_{i+1} > \cdots$ such that

1. $G_{i+1} \triangleleft G_i$.
2. $\bigcap_{i \geq 0} G_i = \{1\}$.
3. $G_i / G_{i+1}$ is an elementary abelian $p$-group.
4. $\ker(G_i \to H_1(G_i, \mathbb{Q})) < G_{i+1}$.

The class of RFR$_p$ groups is closed under taking subgroups, finite direct products, and finite free products.

Finitely generated free groups; closed, orientable surface groups; and right-angled Artin groups are RFR$_p$, for all $p$.

Finite groups and non-abelian nilpotent groups are *not* RFR$_p$, for any $p$. 
**Theorem (Koberda–S. 2016)**

Let $G$ be a f.g. group which is $\text{RFR}_p$ for some prime $p$. Then:

- $G$ is residually finite. In particular, if $G$ is finitely presented, then $G$ has a solvable word problem.
- $G$ is torsion-free.
- $G$ is residually torsion-free polycyclic.

**Theorem**

Let $G$ be a f.p. group which is non-abelian and $\text{RFR}_p$ for infinitely many primes $p$. Then:

- $G$ is bi-orderable.
- The maximal $k$-step solvable quotients $G/G^{(k)}$ are not finitely presented, for any $k \geq 2$.
- $\Sigma^1(G)^\mathbb{C} \neq \emptyset$. 
**LARGE GROUPS**

A finitely generated group $G$ is said to be *large* if there is a finite-index subgroup $H < G$ which surjects onto a free, non-cyclic group.

**Theorem (Koberda 2014)**

An f.p. group $G$ is large if and only if there exists a finite-index subgroup $K < G$ such that $V^1(K)$ has infinitely many torsion points.

**Theorem (KS 2016)**

Let $G$ be a f.p. group which is non-abelian and RFR$p$ for infinitely many primes $p$. Then $G$ is large.

**Proposition (PS 2017, follows from Arapura)**

Let $X$ be a quasi-projective manifold. Then $\pi_1(X)$ is large if and only if there is a finite cover $Y \to X$ and a regular, surjective map from $Y$ to a smooth curve $C$ with $\chi(C) < 0$, so that the generic fiber is connected.
BOUNDARY MANIFOLDS OF PLANE CURVES

- Let \( C \) be a (reduced) algebraic curve in \( \mathbb{CP}^2 \).
- The boundary manifold of \( C \) is defined as \( M_C = \partial T \), where \( T \) is a regular neighborhood of \( C \).
- \( M = M_C \) is a closed, oriented graph-manifold over a graph \( \Gamma \).

**Example**

Suppose \( C \) is smooth. Then \( C \cong \Sigma_g \), where \( g = \binom{d-1}{2} \), and \( d = \deg(C) \). Thus, \( M_C \) is a circle bundle over \( \Sigma_g \) with Euler number \( e = d^2 \).

In this example, \( \pi_1(M) \) is not RFRp, for any prime \( p \), provided \( d \geq 2 \).

**Example**

Suppose \( C = C \cup L \) consists of a smooth conic and a transverse line. The graph \( \Gamma \) is a square, the vertex manifolds are thickened tori \( S^1 \times S^1 \times I \), and \( M_C \) is the Heisenberg nilmanifold.

In this example, \( \pi_1(M) \) is not RFRp, for any prime \( p \).
**Question**

For which plane algebraic curves $C$ is the fundamental group of the boundary manifold $M_C$ an RFR$_p$ group (for some $p$ or all primes $p$)?

**Theorem (KS 2016)**

Let $C$ be an algebraic curve in $\mathbb{C}^2$, with boundary manifold $M$. Suppose that each irreducible component of $C$ is smooth and transverse to the line at infinity, and all singularities of $C$ are of type A. Then $\pi_1(M)$ is RFR$_p$, for all primes $p$.

**Corollary**

If $M$ is the boundary manifold of a line arrangement in $\mathbb{C}^2$, then $\pi_1(M)$ is RFR$_p$, for all primes $p$.

**Conjecture**

Arrangement groups are RFR$_p$, for all primes $p$. 
The lower central series of a group $G$ is defined inductively by $\gamma_1 G = G$ and $\gamma_{k+1} G = [\gamma_k G, G]$.

This forms a filtration of $G$ by characteristic subgroups. The LCS quotients, $\gamma_k G / \gamma_{k+1} G$, are abelian groups.

The group commutator induces a graded Lie algebra structure on $\text{gr}(G, \mathbb{k}) = \bigoplus_{k \geq 1} (\gamma_k G / \gamma_{k+1} G) \otimes \mathbb{Z} \mathbb{k}$.

Assume $G$ is finitely generated. Then $\text{gr}(G)$ is also finitely generated (in degree 1) by $\text{gr}_1(G) = H_1(G, \mathbb{k})$.

For instance, $\text{gr}(F_n)$ is the free graded Lie algebra $\mathbb{L}_n := \text{Lie}(\mathbb{k}^n)$. 
Let \( A \) be a 1-finite cdga. Set \( A_i = (A^i)^* \).

Let \( \mu^*: A_2 \to A_1 \wedge A_1 \) be the dual to the multiplication map \( \mu: A^1 \wedge A^1 \to A^2 \).

Let \( d^*: A_2 \to A_1 \) be the dual of the differential \( d: A^1 \to A^2 \).

The **holonomy Lie algebra** of \( A \) is the quotient

\[
\mathfrak{h}(A) = \text{Lie}(A_1)/\langle \text{im}(\mu^* + d^*) \rangle.
\]

For a f.g. group \( G \), set \( \mathfrak{h}(G) := \mathfrak{h}(H^\bullet(G, \mathbb{k})) \). There is then a canonical surjection \( \mathfrak{h}(G) \to \text{gr}(G) \), which is an isomorphism precisely when \( \text{gr}(G) \) is quadratic.
Let $G$ be a f.g. group. The successive quotients of $G$ by the terms of the LCS form a tower of finitely generated, nilpotent groups,

$$
\cdots \rightarrow G/\gamma_4 G \rightarrow G/\gamma_3 G \rightarrow G/\gamma_2 G = G_{ab}.
$$

(Malcev 1951) It is possible to replace each nilpotent quotient $N_k$ by $N_k \otimes k$, the (rationally defined) nilpotent Lie group associated to the discrete, torsion-free nilpotent group $N_k/\text{tors}(N_k)$.

The inverse limit, $\mathcal{M}(G) = \lim_{\leftarrow k} (G/\gamma_k G) \otimes k$, is a prounipotent, filtered Lie group, called the *prounipotent completion* of $G$ over $k$.

The pronilpotent Lie algebra

$$
\mathfrak{m}(G) := \lim_{\leftarrow k} \text{Lie}((G/\gamma_k G) \otimes k),
$$

endowed with the inverse limit filtration, is called the *Malcev Lie algebra* of $G$ (over $k$).
By dualizing the canonical filtration of $M_1(G)$, we obtain a tower of central extensions of finite-dimensional nilpotent Lie algebras,

$$
\cdots \rightarrow m_{n+1} \rightarrow m_n \rightarrow \cdots \rightarrow m_1 = \{0\};
$$

$m(G)$ is isomorphic to the inverse limit of this tower.

The group-algebra $kG$ has a natural Hopf algebra structure, with comultiplication $\Delta(g) = g \otimes g$ and counit the augmentation map.

(Quillen 1968) The $I$-adic completion of the group-algebra, $\hat{kG} = \lim_k kG/I^k$, is a filtered, complete Hopf algebra.

An element $x \in \hat{kG}$ is called primitive if $\hat{\Delta}x = x\hat{1} + \hat{1}\hat{x}$. The set of all such elements, with bracket $[x, y] = xy - yx$, and endowed with the induced filtration, is a complete, filtered Lie algebra.

We then have $m(G) \cong \text{Prim}(\hat{kG})$ and $\text{gr}(m(G)) \cong \text{gr}(G)$.

(Sullivan 1977) $G$ is 1-formal $\iff m(G)$ is quadratic.
**Finiteness obstructions for groups**

**Lemma**

For $n \geq 2$, the graded vector space $\mathbb{L}''/[[\mathbb{L}, \mathbb{L}'']$ is infinite-dimensional.

**Theorem (PS 2017)**

Let $G$ be a f.g. group which has a free, non-cyclic quotient. Then:

- $G/\mathbb{G}''$ is not finitely presentable.
- $G/\mathbb{G}''$ does not admit a 1-finite 1-model.

**Theorem (PS 2017)**

A f.g. group $G$ admits a 1-finite 1-model $A$ if and only if $m(G)$ is the lcs completion of a finitely presented Lie algebra, namely,

$$m(G) \cong \widehat{h}(A).$$