POINCARÉ DUALITY AND RESONANCE VARIETIES

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Let \( A \) be a graded, graded-commutative algebra over a field \( \mathbb{k} \).

- \( A = \bigoplus_{i \geq 0} A^i \), where \( A^i \) are \( \mathbb{k} \)-vector spaces.
- \( \cdot : A^i \otimes_{\mathbb{k}} A^j \to A^{i+j} \).
- \( ab = (-1)^{ij} ba \) for all \( a \in A^i, b \in A^j \).

We will assume that \( A \) is connected (\( A^0 = \mathbb{k} \cdot 1 \)), and locally finite (all the Betti numbers \( b_i(A) := \dim_\mathbb{k} A^i \) are finite).

\( A \) is a Poincaré duality \( \mathbb{k} \)-algebra of dimension \( m \) if there is a \( \mathbb{k} \)-linear map \( \varepsilon : A^m \to \mathbb{k} \) (called an orientation) such that all the bilinear forms \( A^i \otimes_{\mathbb{k}} A^{m-i} \to \mathbb{k}, a \otimes b \mapsto \varepsilon(ab) \) are non-singular.

That is, \( A \) is a graded, graded-commutative Gorenstein Artin algebra of socle degree \( m \).
If $A$ is a PD$_m$ algebra, then:

- $b_i(A) = b_{m-i}(A)$, and $A^i = 0$ for $i > m$.
- $\varepsilon$ is an isomorphism.
- The maps PD: $A^i \rightarrow (A^{m-i})^\ast$, PD$(a)(b) = \varepsilon(ab)$ are isomorphisms.

Each $a \in A^i$ has a Poincaré dual, $a^\vee \in A^{m-i}$, such that $\varepsilon(aa^\vee) = 1$.

The orientation class is $\omega_A := 1^\vee$.

We have $\varepsilon(\omega_A) = 1$, and thus $aa^\vee = \omega_A$.

The class of $k$-PD algebras is closed under taking tensor products and connected sums.

- If $A$ is PD$_m$ and $B$ is PD$_n$, then $A \otimes_k B$ is PD$_{m+n}$.
- If $A$ and $B$ are PD$_m$, then $A \# B$ is PD$_m$, where

\[
\begin{array}{ccc}
\bigwedge (\omega) & \xrightarrow{\omega \mapsto \omega_A} & A \\
\omega & \downarrow & \\
\omega_B & \downarrow & \\
B & \rightarrow & A \# B
\end{array}
\]
The associated alternating form

Associated to a $\mathbb{k}$-PD$_m$ algebra there is an alternating $m$-form,

$$\mu_A : \bigwedge^m A^1 \to \mathbb{k}, \quad \mu_A(a_1 \wedge \cdots \wedge a_m) = \varepsilon(a_1 \cdots a_m).$$

Assume now that $m = 3$, and set $n = b_1(A)$. Fix a basis $\{e_1, \ldots, e_n\}$ for $A^1$, and let $\{e^1, \ldots, e^n\}$ be the dual basis for $A^2$.

The multiplication in $A$, then, is given on basis elements by

$$e_i e_j = \sum_{k=1}^r \mu_{ijk} e^k, \quad e_i e^j = \delta_{ij}\omega,$$

where $\mu_{ijk} = \mu(e_i \wedge e_j \wedge e_k)$.

Let $A^i = (A^i)^\ast$. We may then view $\mu$ dually as a trivector,

$$\mu = \sum \mu_{ijk} e^i \wedge e^j \wedge e^k \in \bigwedge^3 A_1,$$

which encodes the algebra structure of $A$.

For instance, $\mu_{A\#B} = \mu_A + \mu_B$. 

Classification of Alternating Forms

Let $V$ be a $k$-vector space of dimension $n$. The group $\text{GL}(V)$ acts on $\bigwedge^m(V^*)$ by $(g \cdot \mu)(a_1 \wedge \cdots \wedge a_m) = \mu(g^{-1}a_1 \wedge \cdots \wedge g^{-1}a_m)$.

The orbits of this action are the equivalence classes of alternating $m$-forms on $V$. (We write $\mu \sim \mu'$ if $\mu' = g \cdot \mu$.)

Over $\overline{k}$, the Zariski closures of these orbits define affine algebraic varieties.

There are finitely many orbits over $\overline{k}$ only if $n^2 \geq \binom{n}{m}$, that is, $m \leq 2$ or $m = 3$ and $n \leq 8$.

For $\overline{k} = \mathbb{C}$, each complex orbit has only finitely many real forms.

When $m = 3$, and $n = 8$, there are 23 complex orbits, which split into either 1, 2, or 3 real orbits, for a total of 35 real orbits.
Let $A$ and $B$ be two $\text{PD}_m$ algebras. We say that a morphism of graded algebras $\varphi: A \to B$ has non-zero degree if the linear map $\varphi^m: A^m \to B^m$ is non-zero. (Equivalently, $\varphi$ is injective.)

$A$ and $B$ are isomorphic as $\text{PD}_m$ algebras if and only if they are isomorphic as graded algebras, in which case $\mu_A \sim \mu_B$.

**Proposition**

For two $\text{PD}_3$ algebras $A$ and $B$, the following are equivalent.

1. $A \cong B$, as $\text{PD}_3$ algebras.
2. $A \cong B$, as graded algebras.
3. $\mu_A \sim \mu_B$.

We thus have a bijection between isomorphism classes of 3-dimensional Poincaré duality algebras and equivalence classes of alternating 3-forms, given by $A \leftrightarrow \mu_A$. 
Poincaré duality in orientable manifolds

- If $M$ is a compact, connected, orientable, $m$-dimensional manifold, then the cohomology ring $A = H^*(M, k)$ is a PD$_m$ algebra over $k$.
- Sullivan (1975): for every finite-dimensional $\mathbb{Q}$-vector space $V$ and every alternating 3-form $\mu \in \wedge^3 V^*$, there is a closed 3-manifold $M$ with $H^1(M, \mathbb{Q}) = V$ and cup-product form $\mu_M = \mu$.
- Such a 3-manifold can be constructed via “Borromean surgery.”

- If $M$ bounds an oriented 4-manifold $W$ such that the cup-product pairing on $H^2(W, M)$ is non-degenerate (e.g., if $M$ is the link of an isolated surface singularity), then $\mu_M = 0$. 

![Image of Borromean rings](image.png)
Let $A$ be a graded, graded-commutative, connected, locally finite algebra over a field $k$ (with $\text{char } k \neq 2$).

For each $a \in A^1$ we have $a^2 = -a^2$, and so $a^2 = 0$. We then obtain a cochain complex of $k$-vector spaces,

$$(A, \delta_a) : A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

with differentials $\delta_a^i(u) = a \cdot u$, for all $u \in A^i$.

The resonance varieties of $A$ (in degree $i \geq 0$ and depth $k \geq 0$):

$$\mathcal{R}_k^i(A) = \{ a \in A^1 \mid \dim_k H^i(A, \delta_a) \geq k \}.$$

An element $a \in A^1$ belongs to $\mathcal{R}_k^i(A)$ if and only if there exist $u_1, \ldots, u_k \in A^i$ such that $au_1 = \cdots = au_k = 0$ in $A^{i+1}$, and the set $\{au, u_1, \ldots, u_k\}$ is linearly independent in $A^i$, for all $u \in A^{i-1}$. 
Set \( b_j = b_j(A) \). For each \( i \geq 0 \), we have a descending filtration,

\[
A^i = \mathcal{R}_0^i(A) \supseteq \mathcal{R}_1^i(A) \supseteq \cdots \supseteq \mathcal{R}_{b_i}^i(A) = \{0\} \supseteq \mathcal{R}_{b_{i+1}}^i(A) = \emptyset.
\]

A linear subspace \( U \subset A^1 \) is isotropic if the restriction of \( \cdot : A^1 \wedge A^1 \to A^2 \) to \( U \wedge U \) is the zero map (i.e., \( ab = 0, \forall a, b \in U \)).

If \( U \subset A^1 \) is an isotropic subspace of dimension \( k \), then \( U \subset \mathcal{R}_{k-1}^1(A) \).

\( \mathcal{R}_1^1(A) \) is the union of all isotropic planes in \( A^1 \).

If \( k \subset K \) is a field extension, then the \( k \)-points on \( \mathcal{R}_k^i(A \otimes_k K) \) coincide with \( \mathcal{R}_k^i(A) \).

Let \( \varphi : A \to B \) be a morphism of graded, connected algebras. If the map \( \varphi^1 : A^1 \to B^1 \) is injective, then \( \varphi^1(\mathcal{R}_k^1(A)) \subset \mathcal{R}_k^1(B), \forall k \).
Fix a \( k \)-basis \( \{ e_1, \ldots, e_n \} \) for \( A^1 \), and let \( \{ x_1, \ldots, x_n \} \) be the dual basis for \( A_1 = (A^1)^* \).

Identify \( \text{Sym}(A_1) \) with \( S = k[x_1, \ldots, x_n] \), the coordinate ring of the affine space \( A^1 \).

The Bernstein–Gelfand–Gelfand correspondence yields a cochain complex of finitely generated, free \( S \)-modules, \( L(A) := (A^\bullet \otimes S, \delta) \),

\[
\cdots \rightarrow A^i \otimes S \xrightarrow{\delta^i_A} A^{i+1} \otimes S \xrightarrow{\delta^{i+1}_A} A^{i+2} \otimes S \rightarrow \cdots ,
\]

where \( \delta^i_A(u \otimes s) = \sum_{j=1}^n e_j u \otimes sx_j \).

The specialization of \( (A \otimes S, \delta) \) at \( a \in A^1 \) coincides with \( (A, \delta_a) \), that is, \( \delta^i_A|_{x_j = a_j} = \delta^i_a \).
By definition, an element $a \in A^1$ belongs to $\mathcal{R}_k^i(A)$ if and only if

$$\text{rank } \delta^{i-1}_a + \text{rank } \delta^i_a \leq b_i(A) - k.$$ 

Let $l_r(\psi)$ denote the ideal of $r \times r$ minors of a $p \times q$ matrix $\psi$ with entries in $S$, where $l_0(\psi) = S$ and $l_r(\psi) = 0$ if $r > \min(p, q)$.

Then:

$$\mathcal{R}_k^i(A) = V\left(l_{b_i(A)-k+1}(\delta^{i-1}_A \oplus \delta^i_A)\right)$$

$$= \bigcap_{s+t= b_i(A) - k + 1} \left( V(l_s(\delta^{i-1}_A)) \cup V(l_t(\delta^i_A)) \right).$$

In particular, $\mathcal{R}_k^1(A) = V(l_{n-k}(\delta^1_A))$ ($0 \leq k < n$) and $\mathcal{R}_n^1(A) = \{0\}$.

The (degree $i$, depth $k$) resonance scheme $\mathcal{R}_k^i(A)$ is defined by the ideal $l_{b_i(A)-k+1}(\delta^{i-1}_A \oplus \delta^i_A)$; its underlying set is $\mathcal{R}_k^i(A)$. 
**Example (Exterior algebra)**

Let \( E = \bigwedge V \), where \( V = \mathbb{k}^n \), and \( S = \text{Sym}(V) \). Then \( L(E) \) is the Koszul complex on \( V \). E.g., for \( n = 3 \):

\[
S^3 \overset{(x_2 - x_1 \ 0 \ \ 0 \ x_3 \ 0 \ -x_1 \ 0 \ x_3 \ -x_2)}{\longrightarrow} S^3 \overset{(x_1 \ x_2 \ x_3)}{\longrightarrow} S.
\]

This chain complex provides a free resolution \( L(E) \to \mathbb{k} \) of the trivial \( S \)-module \( \mathbb{k} \). Hence,

\[
\mathcal{R}_k^i(E) = \begin{cases} 
\{0\} & \text{if } k \leq \binom{n}{i}, \\
\emptyset & \text{otherwise}.
\end{cases}
\]
**Example (Non-zero resonance)**

Let \( A = \bigwedge (e_1, e_2, e_3) / \langle e_1 e_2 \rangle \), and set \( S = \mathbb{k}[x_1, x_2, x_3] \). Then

\[
\begin{align*}
\mathbf{L}(A) : & \quad S^2 \xrightarrow{\begin{pmatrix} x_3 & 0 & -x_1 \\ 0 & x_3 & -x_2 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} S.
\end{align*}
\]

\[
\mathcal{R}^1_k(A) = \begin{cases} 
\{x_3 = 0\} & \text{if } k = 1, \\
\{0\} & \text{if } k = 2 \text{ or } 3, \\
\emptyset & \text{if } k > 3.
\end{cases}
\]

**Example (Non-linear resonance)**

Let \( A = \bigwedge (e_1, \ldots, e_4) / \langle e_1 e_3, e_2 e_4, e_1 e_2 + e_3 e_4 \rangle \). Then

\[
\begin{align*}
\mathbf{L}(A) : & \quad S^3 \xrightarrow{\begin{pmatrix} x_4 & 0 & 0 & -x_1 \\ 0 & x_3 & -x_2 & 0 \\ -x_2 & x_1 & x_4 & -x_3 \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}} S.
\end{align*}
\]

\[
\mathcal{R}^1_1(A) = \{x_1 x_2 + x_3 x_4 = 0\}
\]
Resonance varieties of PD-algebras

Let $A$ be a PD$_m$ algebra. For $0 \leq i \leq m$ and $a \in A^1$, the square

$$
(A^{m-i})^* \xrightarrow{\delta^{m-i-1}_{-a}^*} (A^{m-i-1})^*
$$

commutes up to a sign.

Consequently,

$$
\left(H^i(A, \delta_a)\right)^* \cong H^{m-i}(A, \delta_{-a}).
$$

Hence, for all $i$ and $k$,

$$
R^i_k(A) = R^{m-i}_k(A).
$$

In particular, $R^m_1(A) = \{0\}$. 
**Corollary**

Let $A$ be a PD$_3$ algebra with $b_1(A) = n$. Then

1. $R^i_0(A) = A^1$.
2. $R^3_1(A) = R^0_1(A) = \{0\}$ and $R^2_n(A) = R^1_n(A) = \{0\}$.
3. $R^2_k(A) = R^1_k(A)$ for $0 < k < n$.
4. In all other cases, $R^i_k(A) = \emptyset$.

**Theorem**

Every PD$_3$ algebra $A$ decomposes as $A \cong B \# C$, where $B$ are $C$ are PD$_3$ algebras such that $\mu_B$ is irreducible and has the same rank as $\mu_A$, and $\mu_C = 0$. Furthermore, $A^1 \cong B^1 \oplus C^1$ restricts to isomorphisms

$$R^1_k(A) \cong R^1_{k-r+1}(B) \times C^1 \cup R^1_{k-r}(B) \times \{0\} \quad (\forall k \geq 0),$$

where $r = \text{corank} \mu_A$. In particular, $R^1_k(A) = A^1$ for all $k < \text{corank} \mu_A$.

(The rank of a form $\mu : \bigwedge^3 V \to \mathbb{k}$ is the minimum dimension of a linear subspace $W \subset V$ such that $\mu$ factors through $\bigwedge^3 W$.)

A linear subspace $U \subset V$ is 2-singular with respect to a 3-form $\mu : \bigwedge^3 V \to \mathbb{k}$ if $\mu(a \wedge b \wedge c) = 0$ for all $a, b \in U$ and $c \in V$.

If $\dim U = 2$, we simply say $U$ is a singular plane.

The nullity of $\mu$, denoted $\text{null}(\mu)$, is the maximum dimension of a 2-singular subspace $U \subset V$.

Clearly, $V$ contains a singular plane if and only if $\text{null}(\mu) \geq 2$.

Let $A$ be a PD$_3$ algebra. A linear subspace $U \subset A^1$ is 2-singular (with respect to $\mu_A$) if and only if $U$ is isotropic.

Using a result of A. Sikora [2005], we obtain:

**Theorem**

Let $A$ be a PD$_3$ algebra over an algebraically closed field $\mathbb{k}$ ($\text{char}(\mathbb{k}) \neq 2$), and let $\nu = \text{null}(\mu_A)$. If $b_1(A) \geq 4$, then

$$\dim \mathcal{R}_{\nu-1}^1(A) \geq \nu \geq 2.$$ 

In particular, $\dim \mathcal{R}_1^1(A) \geq \nu$. 

Sikora made the following conjecture: If $\mu: \bigwedge^3 V \to k$ is a 3-form with $\dim V \geq 4$ and if $\text{char}(k) \neq 2$, then $\text{null}(\mu) \geq 2$.

Conjecture holds if $n := \dim V$ is even or equal to 5, or if $k = \overline{k}$.

Work of J. Draisma and R. Shaw [2010, 2014] implies that the conjecture does not hold for $k = \mathbb{R}$ and $n = 7$. We obtain:

**Theorem**

Let $A$ be a PD$_3$ algebra over $\mathbb{R}$. Then $\mathcal{R}_1^1(A) \neq \{0\}$, except when

- $n = 1, \mu_A = 0$.
- $n = 3, \mu_A = e_1 e_2 e_3$.
- $n = 7, \mu_A = -e_1 e_3 e_5 + e_1 e_4 e_6 + e_2 e_3 e_6 + e_2 e_4 e_5 + e_1 e_2 e_7 + e_3 e_4 e_7 + e_5 e_6 e_7$.

Sketch: If $\mathcal{R}_1^1(A) = \{0\}$, then the formula $(x \times y) \cdot z = \mu_A(x, y, z)$ defines a cross-product on $A^1 = \mathbb{R}^n$, and thus a division algebra structure on $\mathbb{R}^{n+1}$, forcing $n = 1, 3$ or 7 by Bott–Milnor/Kervaire [1958].
Example

Let $A$ be the real $\text{PD}_3$ algebra corresponding to octonionic multiplication (defined as above).

Let $A'$ be the real $\text{PD}_3$ algebra with 
\[ \mu_{A'} = e^1 e^2 e^3 + e^4 e^5 e^6 + e^1 e^4 e^7 + e^2 e^5 e^7 + e^3 e^6 e^7. \]

Then $\mu_A \sim \mu_{A'}$ over $\mathbb{C}$, and so $A \otimes_{\mathbb{R}} \mathbb{C} \cong A' \otimes_{\mathbb{R}} \mathbb{C}$.

On the other hand, $A \not\cong A'$ over $\mathbb{R}$, since $\mu_A \not\sim \mu_{A'}$ over $\mathbb{R}$, but also because $\mathcal{R}_1^1(A) = \{0\}$, yet $\mathcal{R}_1^1(A') \neq \{0\}$.

Both $\mathcal{R}_1^1(A \otimes_{\mathbb{R}} \mathbb{C})$ and $\mathcal{R}_1^1(A' \otimes_{\mathbb{R}} \mathbb{C})$ are projectively smooth conics, and thus are projectively equivalent over $\mathbb{C}$, but 
\[ \mathcal{R}_1^1(A \otimes_{\mathbb{R}} \mathbb{C}) = \{x \in \mathbb{C}^7 \mid x_1^2 + \cdots + x_7^2 = 0\} \]
has only one real point ($x = 0$), whereas 
\[ \mathcal{R}_1^1(A' \otimes_{\mathbb{R}} \mathbb{C}) = \{x \in \mathbb{C}^7 \mid x_1 x_4 + x_2 x_5 + x_3 x_6 = x_7^2\} \]
contains the real (isotropic) subspace $\{x_4 = x_5 = x_6 = x_7 = 0\}$. 
Pfaffians and Resonance

For a \( k \)-PD3 algebra \( A \), the complex \( L(A) = (A \otimes_k S, \delta_A) \) looks like

\[
A^0 \otimes_k S \xrightarrow{\delta_A^0} A^1 \otimes_k S \xrightarrow{\delta_A^1} A^2 \otimes_k S \xrightarrow{\delta_A^2} A^3 \otimes_k S,
\]

where \( \delta_A^0 = (x_1 \cdots x_n) \) and \( \delta_A^2 = (\delta_A^0)^\top \), while \( \delta_A^1 \) is the skew-symmetric matrix whose entries linear forms in \( S \) given by

\[
\delta_A^1(e_i) = \sum_{j=1}^n \sum_{k=1}^n \mu_{jik} e_k^\vee \otimes x_j.
\]

Recall that \( \mathcal{R}_k^1(A) = V(I_{n-k}(\delta_A^1)) \). Using work of Buchsbaum and Eisenbud [1977] on Pfaffians of skew-symmetric matrices, we get:

**Theorem**

\[
\begin{align*}
\mathcal{R}_{2k}^1(A) &= \mathcal{R}_{2k+1}^1(A) = V(Pf_{n-2k}(\delta_A^1)), & \text{if } n \text{ is even}, \\
\mathcal{R}_{2k-1}^1(A) &= \mathcal{R}_{2k}^1(A) = V(Pf_{n-2k+1}(\delta_A^1)), & \text{if } n \text{ is odd}.
\end{align*}
\]

Hence, \( A^1 = \mathcal{R}_0^1 = \mathcal{R}_1^1 \supsetneq \mathcal{R}_2^1 = \mathcal{R}_3^1 \supsetneq \mathcal{R}_4^1 = \cdots \) if \( b_1(A) \) is even,

and \( A^1 = \mathcal{R}_0^1 \supsetneq \mathcal{R}_1^1 = \mathcal{R}_2^1 \supsetneq \mathcal{R}_3^1 = \mathcal{R}_4^1 \supsetneq \cdots \) if \( b_1(A) \) is odd.
**THEOREM**

Let $A$ be a PD$_3$ algebra. If $\mu_A$ has maximal rank $n \geq 3$, then

$$\mathcal{R}_{n-2}^1(A) = \mathcal{R}_{n-1}^1(A) = \mathcal{R}_n^1(A) = \{0\}.$$ 

Otherwise, write $A = B \neq C$, where $\mu_B$ is irreducible and $\mu_C = 0$. If $n = \dim A^1$ is at least 3, then $\mathcal{R}_{n-2}^1(A) = \mathcal{R}_{n-1}^1(A) = C^1$.

**LEMMA (Turaev 2002)**

Suppose $n \geq 3$. There is then a polynomial $\text{Det}(\mu_A) \in S$ such that, if $\delta_A^1(i; j)$ is the sub-matrix obtained from $\delta_A^1$ by deleting the $i$-th row and $j$-th column, then $\det \delta_A^1(i; j) = (-1)^{i+j} x_i x_j \text{Det}(\mu_A)$.

Moreover, if $n$ is even, then $\text{Det}(\mu_A) = 0$, while if $n$ is odd, then $\text{Det}(\mu_A) = \text{Pf}(\mu_A)^2$, where $\text{pf}(\delta_A^1(i; i)) = (-1)^{i+1} x_i \text{Pf}(\mu_A)$. 
Suppose $\dim_k V = 2g + 1 > 1$. We say that a 3-form $\mu : \bigwedge^3 V \to k$ is generic (in the sense of Berceanu–Papadima [1994]) if there is a $v \in V$ such that the 2-form $\gamma_v \in V^* \wedge V^*$ given by $\gamma_v(a \wedge b) = \mu_A(a \wedge b \wedge v)$ for $a, b \in V$ has rank $2g$, that is, $\gamma_v^g \neq 0$ in $\bigwedge^{2g} V^*$.

**Theorem**

Let $A$ be a PD$_3$ algebra. Then

$$
\mathcal{R}_1(A) = \begin{cases} 
\emptyset & \text{if } n = 0; \\
\{0\} & \text{if } n = 1 \text{ or } n = 3 \text{ and } \mu \text{ has rank } 3; \\
V(\text{Pf}(\mu_A)) & \text{if } n \text{ is odd, } n > 3, \text{ and } \mu_A \text{ is BP-generic}; \\
A^1 & \text{otherwise.}
\end{cases}
$$

**Example**

Let $M = \Sigma_g \times S^1$, where $g \geq 2$. Then $\mu_M = \sum_{i=1}^g a_i b^i c$ is BP-generic, and $\text{Pf}(\mu_M) = x_{2g+1}^{g-1}$. Hence, $\mathcal{R}_1(M) = \{x_{2g+1} = 0\}$. In fact, $\mathcal{R}_1 = \cdots = \mathcal{R}_{2g-2}$ and $\mathcal{R}_{2g-1} = \mathcal{R}_{2g} = \mathcal{R}_{2g+1} = \{0\}$. 
As a corollary, we recover a closely related result, proved by Draisma and Shaw [2010] by very different methods.

**Corollary**

Let $V$ be a $k$-vector space of odd dimension $n \geq 5$ and let $\mu \in \wedge^3 V^*$. Then the union of all singular planes is either all of $V$ or a hypersurface defined by a homogeneous polynomial in $k[V]$ of degree $(n - 3)/2$.

For $\mu \in \wedge^3 V^*$, there is another genericity condition, due to P. De Poi, D. Faenzi, E. Mezzetti, and K. Ranestad [2017]: $\text{rank}(\gamma_v) > 2$, for all non-zero $v \in V$. We may interpret some of their results, as follows.

**Theorem (DFMR)**

Let $A$ be a PD$_3$ algebra over $\mathbb{C}$, and suppose $\mu_A$ is generic. Then:

- If $n$ is odd, then $\mathcal{R}^1_1(A)$ is a hypersurface of degree $(n - 3)/2$ which is smooth if $n \leq 7$, and singular in codimension 5 if $n \geq 9$.
- If $n$ is even, then $\mathcal{R}^1_2(A)$ has codim 3 and degree $\frac{1}{4}(n-2)^3 + 1$; it is smooth if $n \leq 10$, and singular in codimension 7 if $n \geq 12$. 
# Resonance Varieties of 3-forms of Low Rank

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<td>${x_1 = x_2 = x_3 = x_4 = 0} \cup {x_1 = x_4 = x_5 = x_6 = 0}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>123+456+147</td>
<td>$x_1 x_4 + x_2 x_5 = 0$</td>
<td>${x_1 = x_2 = x_4 = x_5 = x_7 = x_8 = 0}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>123+456+147+257</td>
<td>$x_1 x_4 + x_2 x_5 + x_3 x_6 = x_7^2$</td>
<td>${x_1 = x_2 = x_4 = x_5 = x_7 = x_8 = 0}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>147+257+367+358</td>
<td>$C^8$</td>
<td>${x_7 = 0}$</td>
<td>${x_3 = x_5 = x_7 = x_8 = 0} \cup {x_1 = x_3 = x_4 = x_5 = x_7 = 0}$</td>
<td>0</td>
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<tr>
<td></td>
<td>456+147+257+367+358</td>
<td>$C^8$</td>
<td>${x_5 = x_7 = 0}$</td>
<td>${x_3 = x_4 = x_5 = x_7 = x_1 x_8 + x_6^2 = 0}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>123+456+147+358</td>
<td>$C^8$</td>
<td>${x_1 = x_5 = 0} \cup {x_3 = x_4 = 0}$</td>
<td>${x_1 = x_3 = x_4 = x_5 = x_2 x_6 + x_7 x_8 = 0}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>123+456+147+257+358</td>
<td>$C^8$</td>
<td>${x_1 = x_5 = 0} \cup {x_3 = x_4 = x_5 = 0}$</td>
<td>${x_1 = x_2 = x_3 = x_4 = x_5 = x_7 = 0}$</td>
<td>0</td>
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<td>123+456+147+257+367+358</td>
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<td>${x_3 = x_5 = x_1 x_4 - x_7^2 = 0}$</td>
<td>${x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = 0}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>147+268+358</td>
<td>$C^8$</td>
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<td>${x_1 = x_4 = x_7 = x_8 = 0} \cup {x_2 = x_3 = x_5 = x_6 = x_8 = 0}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>147+257+268+358</td>
<td>$C^8$</td>
<td>$L_1 \cup L_2 \cup L_3$</td>
<td>$L_1 \cup L_2$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>456+147+257+268+358</td>
<td>$C^8$</td>
<td>$C_1 \cup C_2$</td>
<td>$L_1 \cup L_2$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>147+257+367+268+358</td>
<td>$C^8$</td>
<td>$L_1 \cup L_2 \cup L_3 \cup L_4$</td>
<td>$L_1' \cup L_2' \cup L_3'$</td>
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</tr>
<tr>
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<td>456+147+257+367+268+358</td>
<td>$C^8$</td>
<td>$C_1 \cup C_2 \cup C_3$</td>
<td>$L_1 \cup L_2 \cup L_3$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>123+456+147+268+358</td>
<td>$C^8$</td>
<td>$C_1 \cup C_2$</td>
<td>$L$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>123+456+147+257+268+358</td>
<td>$C^8$</td>
<td>${f_1 = \cdots = f_{20} = 0}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>123+456+147+257+367+268+358</td>
<td>$C^8$</td>
<td>${g_1 = \cdots = g_{20} = 0}$</td>
<td>0</td>
<td>0</td>
</tr>
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