ALGEBRAIC INVARIANTS OF PURE BRAID-LIKE GROUPS

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Artin’s braid groups

Let $B_n$ be the group of braids on $n$ strings (under concatenation).

$B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

and $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$.

Let $P_n = \ker(B_n \to S_n)$ be the pure braid group on $n$ strings.

$P_n$ is generated by $A_{ij} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$ ($1 \leq i < j \leq n$).
- $B_n = \text{Mod}^1_{0,n}$, the mapping class group of $D^2$ with $n$ marked points.

- Thus, $B_n$ is a subgroup of $\text{Aut}(F_n)$. In fact:

  $$B_n = \{ \beta \in \text{Aut}(F_n) \mid \beta(x_i) = wx_{\tau(i)}w^{-1}, \beta(x_1 \cdots x_n) = x_1 \cdots x_n \}.$$ 

- $P_n$ is a subgroup of $\text{IA}_n = \{ \varphi \in \text{Aut}(F_n) \mid \varphi_* = \text{id} \text{ on } H_1(F_n) \}$.

- A classifying space for $P_n$ is the configuration space

  $$\text{Conf}_n(\mathbb{C}) = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j \}.$$ 

- Thus, $B_n = \pi_1(\text{Conf}_n(\mathbb{C})/S_n)$.

- Moreover, $P_n = F_{n-1} \times_{\alpha_{n-1}} P_{n-1} = F_{n-1} \times \cdots \times F_2 \times F_1$, where $\alpha_n: P_n \subset B_n \hookrightarrow \text{Aut}(F_n)$. 
Welded braid groups

- The set of all permutation-conjugacy automorphisms of $F_n$ forms a subgroup of $wB_n \subset \text{Aut}(F_n)$, called the welded braid group.
- Let $wP_n = \ker(wB_n \to S_n) = \text{IA}_n \cap wB_n$ be the pure welded braid group $wP_n$.
- McCool (1986) gave a finite presentation for $wP_n$. It is generated by the automorphisms $\alpha_{ij} (1 \leq i \neq j \leq n)$ sending $x_i \mapsto x_j x_i x_j^{-1}$ and $x_k \mapsto x_k$ for $k \neq i$, subject to the relations
  \[
  \alpha_{ij} \alpha_{ik} \alpha_{jk} = \alpha_{jk} \alpha_{ik} \alpha_{ij} \quad \text{for } i, j, k \text{ distinct,}
  \]
  \[
  [\alpha_{ij}, \alpha_{st}] = 1 \quad \text{for } i, j, s, t \text{ distinct,}
  \]
  \[
  [\alpha_{ik}, \alpha_{jk}] = 1 \quad \text{for } i, j, k \text{ distinct.}
  \]
The group $wB_n$ (respectively, $wP_n$) is the fundamental group of the space of untwisted flying rings (of unequal diameters), cf. Brendle and Hatcher (2013).

The upper pure welded braid group (or, upper McCool group) is the subgroup $wP_n^+ \subset wP_n$ generated by $\alpha_{ij}$ for $i < j$.

We have $wP_n^+ \cong F_{n-1} \times \cdots \times F_2 \times F_1$.

**Lemma (S.–Wang)**

For $n \geq 4$, the inclusion $wP_n^+ \hookrightarrow wP_n$ admits no splitting.
The virtual braid group $vB_n$ is obtained from $wB_n$ by omitting certain commutation relations.

Let $vP_n = \ker(vB_n \to S_n)$ be the pure virtual braid group.

Bardakov (2004) gave a presentation for $vP_n$, with generators $x_{ij}$ ($1 \leq i \neq j \leq n$), subject to the relations

\[ x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij}, \quad \text{for } i, j, k \text{ distinct,} \]
\[ [x_{ij}, x_{st}] = 1, \quad \text{for } i, j, s, t \text{ distinct.} \]

Let $vP_n^+$ be the subgroup of $vP_n$ generated by $x_{ij}$ for $i < j$. The inclusion $vP_n^+ \hookrightarrow vP_n$ is a split injection.

Bartholdi, Enriquez, Etingof, and Rains (2006) studied $vP_n$ and $vP_n^+$ as groups arising from the Yang-Baxter equation.

They constructed classifying spaces by taking quotients of permutahedra by suitable actions of the symmetric groups.
SUMMARY OF BRAID-LIKE GROUPS
Cohomology rings and Betti numbers

- Arnol’d (1969): \( H^*(P_n) = \bigwedge_{i<j}(e_{ij})/\langle e_{jk}e_{ik} - e_{ij}(e_{ik} - e_{jk}) \rangle \).

- Jensen, McCammond, and Meier (2006):
  \( H^*(wP_n) = \bigwedge_{i\neq j}(e_{ij})/\langle e_{ij}e_{ji}, e_{jk}e_{ik} - e_{ij}(e_{ik} - e_{jk}) \rangle \).

- F. Cohen, Pakhianathan, Vershinin, and Wu (2007):
  \( H^*(wP^+_n) = \bigwedge_{i<j}(e_{ij})/\langle e_{ij}(e_{ik} - e_{jk}) \rangle \).

  \( H^*(vP_n) = \bigwedge_{i\neq j}(e_{ij})/\langle e_{ij}e_{ji}, e_{ij}(e_{ik} - e_{jk}), e_{ji}e_{ik} = (e_{ij} - e_{ik})e_{jk} \rangle \),
  \( H^*(vP^+_n) = \bigwedge_{i<j}(e_{ij})/\langle e_{ij}(e_{ik} - e_{jk}), (e_{ij} - e_{ik})e_{jk} \rangle \).

All these \( \mathbb{Q} \)-algebras \( A \) are quadratic. In fact, they are all Koszul algebras (\( \text{Tor}_i^A(\mathbb{Q}, \mathbb{Q})_j = 0 \) for \( i \neq j \)), except for \( H^*(wP_n) \), \( n \geq 4 \).

- \( P_n \): Kohno (1987).
- \( wP_n \): Conner and Goetz (2015).
- \( wP^+_n \): D. Cohen and G. Pruidze (2008).
- \( vP_n \) and \( vP^+_n \): Bartholdi et al (2006), Lee (2013).
The Betti numbers of the pure-braid like groups are given by

\[
\begin{array}{|c|c|c|c|c|}
\hline
& P_n & wP_n & wP_n^+ & vP_n & vP_n^+ \\
\hline
b_i & s(n, n – i) & \binom{n-1}{i} n^i & s(n, n – i) & L(n, n – i) & S(n, n – i) \\
\hline
\end{array}
\]

Here \( s(n, k) \) are the Stirling numbers of the first kind, \( S(n, k) \) are the Stirling numbers of the second kind, and \( L(n, k) \) are the Lah numbers.
The lower central series of a group $G$ is defined inductively by
$\gamma_1 G = G$ and $\gamma_{k+1} G = [\gamma_k G, G]$.

The group commutator induces a graded Lie algebra structure on $\text{gr}(G) = \bigoplus_{k \geq 1} (\gamma_k G / \gamma_{k+1} G) \otimes \mathbb{Z} \mathbb{Q}$

Assume $G$ is finitely generated. Then $\text{gr}(G)$ is also finitely generated: in degree 1, by $\text{gr}_1(G) = H_1(G, \mathbb{Q})$.

Let $A^* = H^*(G, \mathbb{Q})$, let $\mu_A : A^1 \wedge A^1 \to A^2$ be the cup-product map, and $\mu_A^\vee : A_2 \to A_1 \wedge A_1$ its dual, where $A_i = (A^i)^\vee$.

Define the holonomy Lie algebra $\mathfrak{h}(G) := \mathfrak{h}(A)$ as the quotient $\text{Lie}(A_1)$ by the ideal generated by $\text{im}(\mu_A^\vee) \subset A_1 \wedge A_1 = \text{Lie}_2(A_1)$.

There is a canonical surjection $\mathfrak{h}(G) \twoheadrightarrow \text{gr}(G)$ which is an isomorphism precisely when $\text{gr}(G)$ is quadratic.
Let \( \phi_k(G) = \dim \text{gr}_k(G) \) be the LCS ranks of \( G \).

E.g.: \( \phi_k(F_n) = \frac{1}{k} \sum_{d|k} \mu\left(\frac{k}{d}\right) n^d \).

By the Poincaré–Birkhoff–Witt theorem,

\[
\prod_{k=1}^{\infty} (1 - t^k)^{-\phi_k(G)} = \text{Hilb}\left(U(\text{gr}(G)), t\right). 
\]

**Proposition (Papadima–Yuzvinsky 1999)**

Suppose \( \text{gr}(G) \) is quadratic and \( A = H^*(G; \mathbb{Q}) \) is Koszul. Then \( \text{Hilb}\left(U(\text{gr}(G)), t\right) \cdot \text{Hilb}(A, -t) = 1 \).

Let \( G \) be a pure braid-like group. Then \( \text{gr}(G) \) is quadratic.

Furthermore, if \( G \neq wP_n \ (n \geq 4) \), then \( H^*(G; \mathbb{Q}) \) is Koszul.

Thus,

\[
\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k(G)} = \sum_{i \geq 0} b_i(G)(-t)^i. 
\]
The Chen Lie algebra of a f.g. group $G$ is $\text{gr}(G/ G'')$, the associated graded Lie algebra of its maximal metabelian quotient.

Let $\theta_k(G) = \dim \text{gr}_k(G/ G'')$ be the Chen ranks of $G$.

Easy to see: $\theta_k(G) \leq \phi_k(G)$ and $\theta_k(G) = \phi_k(G)$ for $k \leq 3$.

K.-T. Chen (1951): $\theta_k(F_n) = (k - 1)\binom{n+k-2}{k}$ for $k \geq 2$.

**Theorem (D. Cohen–S. 1993)**

The Chen ranks $\theta_k = \theta_k(P_n)$ are given by $\theta_1 = \binom{n}{2}$, $\theta_2 = \binom{n}{3}$, and $\theta_k = (k - 1)\binom{n+1}{4}$ for $k \geq 3$.

**Corollary**

Let $\Pi_n = F_{n-1} \times \cdots \times F_1$. Then $P_n \not\cong \Pi_n$ for $n \geq 4$, although both groups have the same Betti numbers and LCS ranks.
**Theorem (D. Cohen–Schenck 2015)**

\[
\theta_k(wP_n) = (k - 1)\binom{n}{2} + (k^2 - 1)\binom{n}{3}, \text{ for } k \gg 0.
\]

**Theorem (S.–Wang)**

The Chen ranks \(\theta_k = \theta_k(wP_n^+)\) are given by \(\theta_1 = \binom{n}{2}, \theta_2 = \binom{n}{3}\), and

\[
\theta_k = \sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}, \text{ for } k \geq 3.
\]

**Corollary**

\(wP_n^+ \not\cong P_n\) and \(wP_n^+ \not\cong \Pi_n\) for \(n \geq 4\), although all three groups have the same Betti numbers and LCS ranks.

This answers a question of F. Cohen et al. (2007).
Resonance varieties

- Let $A$ be a graded $\mathbb{C}$-algebra with $A^0 = \mathbb{C}$ and $\dim A^1 < \infty$.
- The (first) resonance variety of $A$ is defined as
  \[ \mathcal{R}_1(A) = \{ a \in A^1 \mid \exists b \in A^1 \setminus \mathbb{C} \cdot a \text{ such that } a \cdot b = 0 \in A^2 \}. \]
- For a finitely generated group $G$, define $\mathcal{R}_1(G) := \mathcal{R}_1(H^*(G; \mathbb{C}))$.
- For instance, $\mathcal{R}_1(F_n) = \mathbb{C}^n$ for $n \geq 2$, and $\mathcal{R}_1(\mathbb{Z}^n) = \{0\}$.

**Proposition (D. Cohen–S. 1999)**

$\mathcal{R}_1(P_n)$ is a union of $\binom{n}{3} + \binom{n}{4}$ linear subspaces of dimension 2.

**Proposition (D. Cohen 2009)**

$\mathcal{R}_1(wP_n)$ is a union of $\binom{n}{2}$ linear subspaces of dimension 2 and $\binom{n}{3}$ linear subspaces of dimension 3.
**Proposition (S.–Wang)**

\[ \mathcal{R}_1(wP_n^+) = \bigcup_{2 \leq i < j \leq n} L_{ij}, \]

where \( L_{ij} \) is a linear subspace of dimension \( i \).

**Lemma (S.–Wang)**

\( \mathcal{R}_1(vP_4^+) \) is the subvariety of \( H^1(vP_4^+, \mathbb{C}) = \mathbb{C}^6 \) defined by

\[
\begin{align*}
    x_{12}x_{24}(x_{13} + x_{23}) + x_{13}x_{34}(x_{12} - x_{23}) - x_{24}x_{34}(x_{12} + x_{13}) &= 0, \\
    x_{12}x_{23}(x_{14} + x_{24}) + x_{12}x_{34}(x_{23} - x_{14}) + x_{14}x_{34}(x_{23} + x_{24}) &= 0, \\
    x_{13}x_{23}(x_{14} + x_{24}) + x_{14}x_{24}(x_{13} + x_{23}) + x_{34}(x_{13}x_{23} - x_{14}x_{24}) &= 0, \\
    x_{12}(x_{13}x_{14} - x_{23}x_{24}) + x_{34}(x_{13}x_{23} - x_{14}x_{24}) &= 0.
\end{align*}
\]
(Quillen 1968) The *Malcev Lie algebra* of a group $G$ is

$$m(G) = \text{Prim}(\hat{Q}G),$$

the primitives in the $I$-adic completion of the group algebra of $G$.

This is a complete, filtered Lie algebra with $\text{gr}(m(G)) \cong \text{gr}(G)$.

A f.g. group $G$ is *1-formal* if its Malcev Lie algebra is quadratic.

Thus, if $G$ is 1-formal, then $G$ is *graded-formal*, i.e., $\text{gr}(G)$ is quadratic.

Conversely, if $G$ is graded-formal and *filtered-formal*, i.e., $m(G) \cong \text{gr}(m(G))$, then $G$ is 1-formal.

Formality properties are preserved under (finite) direct products and free products, and under split injections.
**Theorem (Dimca–Papadima–S. 2009)**

If $G$ is $1$-formal, then $\mathcal{R}_1(G)$ is a union of projectively disjoint, rationally defined linear subspaces of $H^1(G, \mathbb{C})$.

**Theorem (Kohno 1983)**

Fundamental groups of complements of complex projective hypersurfaces (e.g., $F_n$ and $P_n$) are $1$-formal.

**Theorem (Berceanu–Papadima 2009)**

$wP_n$ and $wP_n^+$ are $1$-formal.
**Theorem (S.–Wang)**

$vP_n$ and $vP_n^+$ are 1-formal if and only if $n \leq 3$.

**Proof.**

- There are split monomorphisms

\[ vP_2^+ \rightarrow vP_3^+ \rightarrow vP_4^+ \rightarrow vP_5^+ \rightarrow vP_6^+ \rightarrow \ldots \]

\[ vP_2 \rightarrow vP_3 \rightarrow vP_4 \rightarrow vP_5 \rightarrow vP_6 \rightarrow \ldots \]

- $vP_2^+ = \mathbb{Z}$ and $vP_3^+ \cong \mathbb{Z} \ast \mathbb{Z}^2$. Thus, they are both 1-formal.

- $vP_3 \cong N \ast \mathbb{Z}$ and $P_4 \cong N \times \mathbb{Z}$. Thus, $vP_3$ is 1-formal.

- $R_1(vP_4^+)$ is non-linear. Thus, $vP_4^+$ is not 1-formal.

- Hence, $vP_n^+$ and $vP_n$ ($n \geq 4$) are also not 1-formal.
Formality and Chen Lie algebras

**Theorem (S–Wang)**

Let $G$ be a finitely generated group. The quotient map $G \rightarrow G/G''$ induces a natural epimorphism of graded Lie algebras,

$$\text{gr}(G)/\text{gr}(G)'' \rightarrow \text{gr}(G/G'').$$

Moreover, if $G$ is filtered-formal, this map is an isomorphism.


There is a natural epimorphism of graded Lie algebras,

$$\mathfrak{g}(G)/\mathfrak{g}(G)'' \rightarrow \text{gr}(G/G'').$$

Moreover, if $G$ is 1-formal, then this map is an isomorphism.

Hence, if $A = H^*(G, \mathbb{Q})$, and $\theta_k(A) := \dim \mathfrak{g}(A)/\mathfrak{g}(A)''$, then $\theta_k(A) \geq \theta_k(G)$, with equality if $G$ is 1-formal.
**The resonance Chen ranks formula**

**Conjecture (S. 2001)**

Let $G$ be a hyperplane arrangement group. Let $c_m(G)$ be the number of $m$-dimensional components of $\mathcal{R}_1(G)$. Then, for $k \gg 1$,

$$\theta_k(G) = \sum_{m \geq 2} c_m(G) \cdot \binom{m + k - 2}{k}.$$

- The conjecture was based in part on $\theta_k(P_n)$ versus $\mathcal{R}_1(P_n)$.
- The inequality $\geq$ was proved in [Schenck–S, 2006], using the 1-formality of arrangement groups.

**Theorem (D. Cohen–Schenck 2015)**

More generally, the conjecture holds if $G$ is a 1-formal, commutator-relators group for which the components of $\mathcal{R}_1(G)$ are isotropic, projectively disjoint, and reduced (as schemes).
**Theorem (S.–Wang)**

Let $A$ be a graded algebra with $\dim A^1 \leq \infty$. Suppose that all the irreducible components of the first resonance variety $\mathcal{R}_1(A)$ are linear, isotropic, and pairwise projectively disjoint. Then, for all $k \gg 0$,

$$
\theta_k(A) \geq (k - 1) \sum_{m \geq 2} \binom{m + k - 2}{k} c_m(A).
$$

Furthermore, if each irreducible component of $\mathcal{R}_1(A)$ is reduced, then equality holds for $k \gg 0$.

- For $A = H^*(G, \mathbb{C})$, this theorem recovers that of Cohen and Schenck, without the commutator-relators assumption.
- The groups $wP_n$ satisfy the Chen ranks formula.
- However, $wP_n^+$ does not satisfy the Chen ranks formula for $n \geq 4$. (The components of $\mathcal{R}_1(wP_n^+)$ are linear and projectively disjoint, but they are neither isotropic, nor reduced).


