FORMALITY NOTIONS FOR SPACES AND GROUPS

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Let $A = (A^\bullet, d)$ be a commutative, differential graded algebra over a field $\mathbb{k}$ of characteristic 0.

1. $A = \bigoplus_{i \geq 0} A^i$, where $A^i$ are $\mathbb{k}$-vector spaces.
2. The multiplication $\cdot : A^i \otimes A^j \to A^{i+j}$ is graded-commutative, i.e., $ab = (-1)^{|a||b|}ba$ for all homogeneous $a$ and $b$.
3. The differential $d : A^i \to A^{i+1}$ satisfies the graded Leibnitz rule, i.e., $d(ab) = d(a)b + (-1)^{|a|}a d(b)$.

The cohomology $H^\bullet(A)$ of the cochain complex $(A, d)$ inherits an algebra structure from $A$.

A cdga morphism $\varphi : A \to B$ is both an algebra map and a cochain map. Hence, $\varphi$ induces a morphism $\varphi^* : H^\bullet(A) \to H^\bullet(B)$.

The map $\varphi$ is a quasi-isomorphism if $\varphi^*$ is an isomorphism. Likewise, $\varphi$ is a $q$-quasi-isomorphism (for some $q \geq 1$) if $\varphi^*$ is an isomorphism in degrees $\leq q$ and is injective in degree $q + 1$. 
Two cdgas, $A$ and $B$, are weakly ($q$-)equivalent ($\simeq_q$) if there is a zig-zag of ($q$-)quasi-isomorphisms connecting $A$ to $B$.

(Sullivan 1977) A cdga $(A, d)$ is formal (or just $q$-formal) if it is ($q$-)weakly equivalent to $(H^\bullet(A), d = 0)$.

Formality implies uniform vanishing of all Massey products.

E.g., if $A$ is 1-formal, then all triple Massey products in $H^2(A)$ must vanish modulo indeterminancy: if $a, b, c \in H^1(A)$, and $ab = bc = 0$, then $\langle a, b, c \rangle = 0$ in $H^\bullet(A)/(a, c)$.

(Halperin–Stasheff 1979) Let $K/k$ be a field extension. A $k$-cdga $(A, d)$ with $H^\bullet(A)$ of finite-type is formal if and only if the $K$-cdga $(A \otimes K, d \otimes \text{id}_K)$ is formal.

(S.–He Wang 2015) Suppose $\dim H^{\leq q+1}(A) < \infty$ and $H^0(A) = k$. Then $(A, d)$ is $q$-formal iff $(A \otimes K, d \otimes \text{id}_K)$ is $q$-formal.
To a large extent, the rational homotopy type of a space can be reconstructed from algebraic models associated to it.

If the space is a smooth manifold $M$, the standard $\mathbb{R}$-model is the de Rham algebra $\Omega_{dR}(M)$.

More generally, any (path-connected) space $X$ has an associated Sullivan $\mathbb{Q}$-cdga, $A_{PL}(X)$. In particular, $H^\bullet(A_{PL}(X)) = H^\bullet(X, \mathbb{Q})$.

An *algebraic (q-)model* (over $k$) for $X$ is a $k$-cgda $(A, d)$ which is (q-) weakly equivalent to $A_{PL}(X) \otimes_{\mathbb{Q}} k$.

For instance, every smooth, quasi-projective variety $X$ admits a finite-dimensional, rational model $A = A(X, D)$, constructed by Morgan from a normal-crossings compactification $\overline{X} = X \cup D$. 
A space $X$ is *(q-)*formal if $A_{PL}(X)$ has this property, i.e.,
$(H^{\bullet}(X, \mathbb{Q}), d = 0)$ is a (q-)model for $X$.

Spheres, Lie groups and their classifying spaces, homogeneous spaces $G/K$ with $rkG = rkK$, and $K(\pi, n)$’s with $n \geq 2$ are formal.

Formality is preserved under (finite) direct products and wedges of spaces, as well as connected sums of manifolds.

The 1-formality property of $X$ depends only on $\pi_1(X)$.

(Macinic 2010) If $X$ is a $q$-formal CW-complex of dimension at most $q + 1$, then $X$ is formal.

A Koszul algebra is a graded $k$-algebra such that $\text{Tor}_s^A(k, k)_t = 0$ for all $s \neq t$.

(Papadima–Yuzvinsky 1999) Suppose $H^{\bullet}(X, k)$ is a Koszul algebra. Then $X$ is formal if and only if $X$ is 1-formal.
(Stasheff 1983) Let $X$ be a $k$-connected CW-complex of dimension $n$. If $n \leq 3k + 1$, then $X$ is formal.

(Miller 1979) If $M$ is a closed, $k$-connected manifold of dimension $n \leq 4k + 2$, then $M$ is formal. In particular, all simply-connected, closed manifolds of dimension at most 6 are formal.

(Fernández–Muñoz 2004) There exist closed, simply-connected, non-formal manifolds of dimension 7.

(Deligne–Griffiths–Morgan–Sullivan 1975) All compact Kähler manifolds are formal.

(Papadima–S. 2015) If $M$ is a compact Sasakian manifold of dimension $2n + 1$, then $M$ is $(2n - 1)$-formal.
Purity implies formality

(Morgan 1978) Let $X$ be a smooth, quasi-projective variety. If $W_1 H^1(X, \mathbb{C}) = 0$, then $X$ is 1-formal.

(Dupont 2016) More generally, suppose either
- $H^k(X)$ is pure of weight $k$, for all $k \leq q + 1$, or
- $H^k(X)$ is pure of weight $2k$, for all $k \leq q$.

Then $X$ is $q$-formal.

In particular, complements of hypersurfaces in $\mathbb{CP}^n$ are 1-formal. Thus, complements of plane algebraic curves are formal.

Complements of linear and toric arrangements are formal, but complements of elliptic arrangements may be non-1-formal.
Resonance varieties of a CDGA

- Assume the cdga \((A, d)\) is connected, i.e., \(A^0 = \mathbb{k}\), and of finite-type, i.e., \(\dim A^i < \infty\) for all \(i \geq 0\).
- For each \(a \in Z^1(A) \cong H^1(A)\), we have a cochain complex,

\[
\begin{array}{cccc}
(A^\bullet, \delta_a) : & A^0 & \xrightarrow{\delta^0_a} & A^1 & \xrightarrow{\delta^1_a} & A^2 & \xrightarrow{\delta^2_a} & \cdots,
\end{array}
\]

with differentials \(\delta^i_a(u) = a \cdot u + d u\), for all \(u \in A^i\).
- The resonance varieties of \((A, d)\) are the sets

\[
\mathcal{R}^i(A) = \{a \in H^1(A) \mid H^i(A^\bullet, \delta_a) \neq 0\}.
\]
- An element \(a \in H^1(A)\) belongs to \(\mathcal{R}^i(A)\) if and only if

\[
\text{rank } \delta^i_a + 1 + \text{rank } \delta^i_a < b_i(A).
\]
- If \(d = 0\), then the resonance varieties of \(A\) are homogeneous.
The *resonance varieties* of a connected, finite-type CW-complex $X$ are the subsets $\mathcal{R}^i(X) := \mathcal{R}^i(H^\bullet(X, \mathbb{C}), d = 0)$ of $H^1(X, \mathbb{C})$.

The variety $\mathcal{R}^1(X)$ depends only on the group $G = \pi_1(X)$; in fact, only on the second nilpotent quotient $G/\gamma_3(G)$.

The *characteristic varieties* of $X$ are the Zariski closed sets of the character group of $G$ given by

$$\mathcal{V}^i(X) = \{ \rho \in \text{Hom}(G, \mathbb{C}^*) \mid H^i(X, \mathbb{C}_\rho) \neq 0 \}.$$  

The variety $\mathcal{V}^1(X)$ depends only on the group $G = \pi_1(X)$; in fact, only on the second derived quotient $G/G''$.

Given any subvariety $W \subset (\mathbb{C}^*)^n$, there is a finitely presented group $G$ such that $G_{ab} = \mathbb{Z}^n$ and $\mathcal{V}^1(G) = W$. 
The Tangent Cone theorem


\[ \tau_1(\mathcal{V}^i(X)) \subseteq TC_1(\mathcal{V}^i(X)) \subseteq R^i(X). \]

- Here, if \( W \subset (\mathbb{C}^*)^n \) is an algebraic subset, then

\[ \tau_1(W) := \{ z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \text{ for all } \lambda \in \mathbb{C} \} \]

is a finite union of rationally defined linear subspaces of \( \mathbb{C}^n \).

- (DPS 2009/DP 2014) If \( X \) is \( q \)-formal, then, for all \( i \leq q \),

\[ \tau_1(\mathcal{V}^i(X)) = TC_1(\mathcal{V}^i(X)) = R^i(X). \]

- This theorem yields a very efficient formality test.
**Example**

Let $G = \langle x_1, x_2 \mid [x_1, [x_1, x_2]] \rangle$. Then $V^1(\pi) = \{ t_1 = 1 \}$, and so $TC_1(V^1(\pi)) = \{ x_1 = 0 \}$. But $R^1(\pi) = \mathbb{C}^2$, and so $\pi$ is not 1-formal.

**Example**

Let $G = \langle x_1, \ldots, x_4 \mid [x_1, x_2], [x_1, x_4][x_2^{-2}, x_3], [x_1^{-1}, x_3][x_2, x_4] \rangle$. Then $R^1(\pi) = \{ z \in \mathbb{C}^4 \mid z_1^2 - 2z_2^2 = 0 \}$: a quadric which splits into two linear subspaces over $\mathbb{R}$, but is irreducible over $\mathbb{Q}$. Thus, $\pi$ is not 1-formal.

**Example**

Let $Conf_n(E)$ be the configuration space of $n$ labeled points of an elliptic curve. Then

$$R^1(Conf_n(E)) = \left\{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \mid \begin{array}{c} \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0, \\ x_iy_j - x_jy_i = 0, \text{ for } 1 \leq i < j < n \end{array} \right\}.$$

For $n \geq 3$, this is an irreducible, non-linear variety (a rational normal scroll). Hence, $Conf_n(E)$ is not 1-formal.
The lower central series of a group $G$ is defined inductively by $\gamma_1 G = G$ and $\gamma_{k+1} G = [\gamma_k G, G]$.

This forms a filtration of $G$ by characteristic subgroups. The LCS quotients, $\gamma_k G / \gamma_{k+1} G$, are abelian groups.

The group commutator induces a graded Lie algebra structure on $\text{gr}(G, \mathbb{k}) = \bigoplus_{k \geq 1} (\gamma_k G / \gamma_{k+1} G) \otimes_{\mathbb{Z}} \mathbb{k}$.

Assume $G$ is finitely generated. Then $\text{gr}(G, \mathbb{k})$ is also finitely generated (in degree 1) by $\text{gr}_1(G, \mathbb{k}) = H_1(G, \mathbb{k})$.

For instance, if $F_n$ is the free group of rank $n$, then $\text{gr}(F_n; \mathbb{k})$ is the free graded Lie algebra $\text{Lie}(\mathbb{k}^n)$. 
Holonomy Lie algebras

- Let $A$ be a commutative graded algebra with $A^0 = \mathbb{k}$ and $\dim A^1 < \infty$. Set $A_i = (A^i)^*$.  

- The multiplication map $A^1 \otimes_\mathbb{k} A^1 \to A^2$ factors through a linear map $\mu_A : A^1 \wedge A^1 \to A^2$.

- Dualizing, and identifying $(A^1 \wedge A^1)^* \cong A_1 \wedge A_1$, we obtain a linear map, $\mu_A^* : A_2 \to A_1 \wedge A_1 \cong \text{Lie}_2(A_1)$.

- The holonomy Lie algebra of $A$ is the quotient
  \[ \mathfrak{h}(A) = \text{Lie}(A_1)/\langle \text{im } \mu_A^* \rangle. \]

- $\mathfrak{h}(A)$ is a quadratic Lie algebra, which depends only on the quadratic closure, $\bar{A} := \wedge(A^1)/\langle \text{ker } \mu_A \rangle$. In fact, $U(\mathfrak{h}(A)) = \bar{A}_!$.

- For a f.g. group $G$, set $\mathfrak{h}(G, \mathbb{k}) := \mathfrak{h}(H^\bullet(G, \mathbb{k}))$. There is then a canonical surjection $\mathfrak{h}(G, \mathbb{k}) \twoheadrightarrow \text{gr}(G, \mathbb{k})$, which is an isomorphism precisely when $\text{gr}(G, \mathbb{k})$ is quadratic.
Let $G$ be a f.g. group. The successive quotients of $G$ by the terms of the LCS form a tower of finitely generated, nilpotent groups,

$$
\cdots \longrightarrow G/\gamma_4 G \longrightarrow G/\gamma_3 G \longrightarrow G/\gamma_2 G = G_{ab}.
$$

(Malcev 1951) It is possible to replace each nilpotent quotient $N_k$ by $N_k \otimes k$, the (rationally defined) nilpotent Lie group associated to the discrete, torsion-free nilpotent group $N_k/\text{tors}(N_k)$.

The inverse limit, $\mathcal{M}(G; k) = \lim_k (G/\gamma_k G) \otimes k$, is a prounipotent, filtered Lie group, called the prounipotent completion of $G$ over $k$.

The pronilpotent Lie algebra

$$
\mathfrak{m}(G; k) := \lim_k \text{Lie}((G/\gamma_k G) \otimes k),
$$

endowed with the inverse limit filtration, is called the Malcev Lie algebra of $G$ (over $k$).
The group-algebra \( \mathbb{k}G \) has a natural Hopf algebra structure, with comultiplication \( \Delta(g) = g \otimes g \) and counit the augmentation map.

(Quillen 1968) The \( I \)-adic completion of the group-algebra, \( \hat{\mathbb{k}G} = \varprojlim_k \mathbb{k}G/I^k \), is a filtered, complete Hopf algebra.

An element \( x \in \hat{\mathbb{k}G} \) is called \textit{primitive} if \( \hat{\Delta}x = x \hat{\otimes} 1 + 1 \hat{\otimes} x \). The set of all such elements, with bracket \([x, y] = xy - yx\), and endowed with the induced filtration, is a complete, filtered Lie algebra.

We then have

\[
m(G) \cong \text{Prim}(\hat{\mathbb{k}G}).
\]

\[
gr(m(G)) \cong gr(G).
\]

(Sullivan 1977) The group \( G \) is 1-formal if and only if its Malcev Lie algebra is quadratic.
**Graded and filtered formality**

- The group $G$ is *graded-formal* if its associated graded Lie algebra $\text{gr}(G)$ is quadratic.

- The group $G$ is *filtered formal* if its Malcev Lie algebra is filtered formal, i.e.,

$$m(G) \cong \text{gr}(m(G))$$

- $G$ is 1-formal $\iff$ $G$ is both graded-formal and filtered-formal.

- The group $G = \langle x_1, x_2 \mid [x_1, [x_1, x_2]] = 1 \rangle$ is filtered-formal. Yet $G$ has a non-trivial 3MP of the form $\langle x_1, x_1, x_2 \rangle$. Hence, $G$ is not graded-formal.

- The group $G = \langle x_1, \ldots, x_5 \mid [x_1, x_2][x_3, [x_4, x_5]] = 1 \rangle$ is graded-formal. Yet $G$ has a non-trivial 3MP of the form $\langle x_3, x_4, x_5 \rangle$. Hence, $G$ is not filtered-formal.
**Theorem (S.–Wang 2015)**

Let $H \leq G$ be a subgroup which admits a split monomorphism $H \to G$. If $G$ is graded-/filtered-/$\mathcal{F}$-formal then $H$ is graded-/filtered-/$\mathcal{F}$-formal.

**Theorem (SW)**

Let $G_1$ and $G_2$ be two f.g. groups. TFAE:

- $G_1$ and $G_2$ are graded-/filtered-/$\mathcal{F}$-formal.
- $G_1 \ast G_2$ is graded-/filtered-/$\mathcal{F}$-formal.
- $G_1 \times G_2$ is graded-/filtered-/$\mathcal{F}$-formal.
Suppose $\varphi : G_1 \to G_2$ is a homomorphism between two f.g. groups, inducing an isomorphism $H_1(G_1; k) \to H_1(G_2; k)$ and an epimorphism $H_2(G_1; k) \to H_2(G_2; k)$. Then:

- If $G_2$ is $1$-formal, then $G_1$ is also $1$-formal.
- If $G_2$ is filtered-formal, then $G_1$ is also filtered-formal.
- If $G_2$ is graded-formal, then $G_1$ is also graded-formal.

Let $K/k$ be a field extension. A f.g. group $G$ is graded-/filtered-/1-formal over $k$ if and only if $G$ is graded-/filtered-/1-formal over $K$. 
Expansions in groups

- Let $\text{gr}(\mathbb{k} G)$ be the associated graded algebra of $\mathbb{k} G$ with respect to the augmentation ideal, and let $\hat{\text{gr}}(\mathbb{k} G)$ be its degree completion.

- (D. Bar-Natan) A *multiplicative expansion* of a group $G$ is a map

\[ E : G \rightarrow \hat{\text{gr}}(\mathbb{k} G) \]

such that the induced algebra morphism, $\bar{E} : \mathbb{k} G \rightarrow \hat{\text{gr}}(\mathbb{k} G)$, is filtration-preserving and induces the identity on associated graded algebras.

- Such a map $E$ is called a *Taylor expansion* if it sends all elements of $G$ to group-like elements of the Hopf algebra $\hat{\text{gr}}(\mathbb{k} G)$.
• $G$ is said to be **residually torsion-free nilpotent** if any non-trivial element of $G$ can be detected in a torsion-free nilpotent quotient.

• If $G$ is finitely generated, the RTFN condition is equivalent to the injectivity of the canonical map $G \to \mathcal{M}(G, \mathbb{k})$.

**Theorem (SW)**

Let $G$ be a finitely generated group. Then:

• $G$ is filtered-formal iff $G$ has a Taylor expansion $G \to \hat{gr}(kG)$.

• $G$ is 1-formal iff $G$ has a Taylor expansion and $gr(kG)$ is a quadratic algebra.

• $G$ has an injective Taylor expansion iff $G$ is residually torsion-free nilpotent and filtered-formal.
(Hasegawa 1989) A nilmanifold $M^n$ is formal iff $M$ is an $n$-torus.

Let $G$ be a finitely generated nilpotent group.

- (Macinic–Papadima 2007) $\mathcal{V}^i(G) \subseteq \{1\}$.

- (Macinic 2010) If $G$ is $q$-formal, then $H^{\leq q+1}(G, k)$ is generated by $H^1(G, k)$. The converse holds if $G$ is 2-step nilpotent.

Let $G$ be a finitely generated, torsion-free, nilpotent group.

- (Carlson–Toledo 1995, Plantiko 1996) Suppose there is a non-zero decomposable element in the kernel of $\cup : H^1(G, k) \wedge H^1(G, k) \rightarrow H^2(G, k)$; then $G$ is not graded-formal.

- (SW) Suppose $G$ is filtered-formal. Then $G$ is abelian if and only if $U(\text{gr}(G, k))$ is Koszul.

- (SW) If $G$ is 2-step nilpotent, and $G_{ab}$ is torsion-free, then $G$ is filtered-formal.
**Theorem (SW)**

Let $G$ be a finitely generated, filtered-formal group. Then all the nilpotent quotients $G/\gamma_i(G)$ are filtered-formal.

- Consequently, all the $n$-step, free nilpotent groups $F_k/\gamma_n F_k$ are filtered-formal.

- The unipotent groups $U_n(\mathbb{Z})$ of integer, upper triangular $n \times n$ matrices with 1’s along the diagonal are filtered-formal, but not graded-formal for $n \geq 3$.

- All nilpotent Lie algebras of dimension 4 or less are filtered-formal (or, “Carnot”).

- (Cornulier 2016) There is a 5-dimensional, 3-step nilpotent Lie algebra which is not filtered-formal.
**Theorem (SW)**

Let $G$ be a finitely generated group. For each $i \geq 2$, the quotient map $G \to G/G^{(i)}$ induces a natural epimorphism of graded $\mathbb{k}$-Lie algebras,

$$\text{gr}(G, \mathbb{k})/\text{gr}(G, \mathbb{k})^{(i)} \to \text{gr}(G/G^{(i)}, \mathbb{k}).$$

Moreover,

- If $G$ is filtered-formal, then each solvable quotient $G/G^{(i)}$ is also filtered-formal, and the above map is an isomorphism.

- If $G$ is 1-formal, then $\mathfrak{h}(G, \mathbb{k})/\mathfrak{h}(G, \mathbb{k})^{(i)} \cong \text{gr}(G/G^{(i)}, \mathbb{k}).$
The quotient map $G \to G/G''$ induces a natural epimorphism of graded Lie algebras,

$$\text{gr}(G, k)/\text{gr}(G, k)'' \to \text{gr}(G/G'', k).$$

Moreover, if $G$ is filtered-formal, this map is an isomorphism.

There is a natural epimorphism of graded Lie algebras,

$$\mathfrak{h}(G, k)/\mathfrak{h}(G, k)'' \to \text{gr}(G/G'', k).$$

Moreover, if $G$ is 1-formal, then this map is an isomorphism.