Geometry and topology of cohomology jumping loci

Alex Suciu
Northeastern University

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Characteristic varieties

- $X$ connected CW-complex with finite $k$-skeleton ($k \geq 1$)
- $G = \pi_1(X, x_0)$: a finitely generated group
- $\text{Hom}(G, \mathbb{C}^\times)$ character variety

**Definition**

For $0 \leq i \leq k$ and $d > 0$, set

$$\mathcal{V}_d^i(X) = \{ \rho \in \text{Hom}(G, \mathbb{C}^\times) \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geq d \},$$

where $\mathbb{C}_\rho$ is the rank 1 local system defined by $\rho$, i.e., $\mathbb{C}$ viewed as a module over $\mathbb{Z}G$, via $g \cdot x = \rho(g)x$, and $H_i(X, \mathbb{C}_\rho) = H_i(C_*(\tilde{X}) \otimes_{\mathbb{Z}G} \mathbb{C}_\rho)$.

- For each $i$, get stratification $\text{Hom}(G, \mathbb{C}^\times) \supseteq \mathcal{V}_1^i \supseteq \mathcal{V}_2^i \supseteq \cdots$
- Note: at $\rho = 1$, $H_i(X, \mathbb{C}_\rho) = H_i(X, \mathbb{C})$. Thus,
  $$1 \in \mathcal{V}_1^i(X) \iff b_i(X) \neq 0$$
- Note: $\mathcal{V}_d(X) = \mathcal{V}^1_d(X)$ depends only on $G$. Write it as $\mathcal{V}_d(G)$. 
Example (Circle)

We have $\widetilde{S^1} = \mathbb{R}$.
Identify $\pi_1(S^1, \ast) = \mathbb{Z} = \langle t \rangle$ and $\mathbb{Z} \mathbb{Z} = \mathbb{Z}[t^{\pm 1}]$. Then:

$$C_\ast(\widetilde{S^1}) : 0 \longrightarrow \mathbb{Z}[t^{\pm 1}] \xrightarrow{t^{-1}} \mathbb{Z}[t^{\pm 1}] \longrightarrow 0$$

For $\rho \in \text{Hom}(\mathbb{Z}, \mathbb{C}^\times) = \mathbb{C}^\times$, get

$$C_\ast(\widetilde{S^1}) \otimes_{\mathbb{Z} \mathbb{Z}} \mathbb{C}_\rho : 0 \longrightarrow \mathbb{C} \xrightarrow{\rho^{-1}} \mathbb{C} \longrightarrow 0$$

which is exact, except for $\rho = 1$, when $H_0(S^1, \mathbb{C}) = H_1(S^1, \mathbb{C}) = \mathbb{C}$. Hence:

$$\mathcal{V}^0_1(S^1) = \mathcal{V}^1_1(S^1) = \{1\}$$
$$\mathcal{V}^i_d(S^1) = \emptyset, \text{ otherwise.}$$
**Example (Torus)**

Identify $\pi_1(T^n) = \mathbb{Z}^n$, and $\text{Hom}(\mathbb{Z}^n, \mathbb{C}^\times) = (\mathbb{C}^\times)^n$. Then:

$$V^i_d(T^n) = \begin{cases} \{1\} & \text{if } d \leq \binom{n}{i}, \\ \emptyset & \text{otherwise}. \end{cases}$$

**Example (Wedge of circles)**

Identify $\pi_1(\bigvee^n S^1) = F_n$, and $\text{Hom}(F_n, \mathbb{C}^\times) = (\mathbb{C}^\times)^n$. Then:

$$V^1_d(\bigvee^n S^1) = \begin{cases} (\mathbb{C}^\times)^n & \text{if } d < n, \\ \{1\} & \text{if } d = n, \\ \emptyset & \text{if } d > n. \end{cases}$$

**Example (Orientable surface of genus $g > 1$)**

$$V^1_d(\Sigma_g) = \begin{cases} (\mathbb{C}^\times)^{2g} & \text{if } d < 2g - 1, \\ \{1\} & \text{if } d = 2g - 1, 2g, \\ \emptyset & \text{if } d > 2g. \end{cases}$$
Alexander polynomial

- $G = \pi_1(X, x_0)$
- $X^{ab} \xrightarrow{p} X$ maximal torsion-free abelian cover, defined by $G^{ab} \xrightarrow{} H = H_1(G) / \text{tors} \cong \mathbb{Z}^n$
- $A_G = H_1(X^{ab}, p^{-1}(x_0); \mathbb{Z})$ Alex. module / $\mathbb{Z}H \cong \mathbb{Z}[t_1^{±1}, \ldots, t_n^{±1}]$
- $\Delta_G = \gcd(E_1(A_G))$

Proposition (Dimca–Papadima–S.)

$$\tilde{V}_1(G) \setminus \{1\} = V(\Delta_G) \setminus \{1\},$$

where

- $\tilde{V}_1(G) = \text{union of codim. 1 components of } V_1(G) \cap \text{Hom}(G, \mathbb{C}^\times)^0$
- $V(\Delta_G) = \text{hypersurface in } \text{Hom}(G, \mathbb{C}^\times)^0 \text{ defined by } \Delta_G$.

Example

Let $K$ be a non-trivial knot, $G = \pi_1(S^3 \setminus K)$. Then:

$$V_1(G) = \{z \in \mathbb{C} \mid \Delta_G(z) = 0\} \cup \{1\}.$$
Tangent cones and exponential tangent cones

The homomorphism $\mathbb{C} \rightarrow \mathbb{C}^\times$, $z \mapsto e^z$ induces

$$\exp: \text{Hom}(G, \mathbb{C}) \rightarrow \text{Hom}(G, \mathbb{C}^\times), \quad \exp(0) = 1$$

Let $W = V(I)$ be a Zariski closed subset in $\text{Hom}(G, \mathbb{C}^\times)$.

**Definition**

- The *tangent cone* at 1 to $W$:

  $$TC_1(W) = V(\text{in}(I))$$

- The *exponential tangent cone* at 1 to $W$:

  $$\tau_1(W) = \{ z \in \text{Hom}(G, \mathbb{C}) \mid \exp(tz) \in W, \forall t \in \mathbb{C} \}$$
Both types of tangent cones
- are homogeneous subvarieties of $\text{Hom}(G, \mathbb{C})$
- are non-empty iff $1 \in W$
- depend only on the analytic germ of $W$ at 1
- commute with finite unions and arbitrary intersections

Moreover,
- $\tau_1(W) \subseteq TC_1(W)$
  - $= \text{ if all irreducible components of } W \text{ are subtori}$
  - $\neq \text{ in general}$
- $\tau_1(W)$ is a finite union of rationally defined subspaces
Bieri–Neumann–Strebel–Renz invariants

$G$ finitely generated group $\leadsto C(G)$ Cayley graph.

$\chi : G \to \mathbb{R}$ homomorphism $\leadsto C_\chi(G)$ induced subgraph on vertex set $G_\chi = \{ g \in G \mid \chi(g) \geq 0 \}$.

**Definition**

$\Sigma^1(G) = \{ \chi \in \text{Hom}(G, \mathbb{R}) \setminus \{0\} \mid C_\chi(G) \text{ is connected} \}$

An open, conical subset of $\text{Hom}(G, \mathbb{R}) = H^1(G, \mathbb{R})$, independent of choice of generating set for $G$.

**Definition**

$\Sigma^k(G, \mathbb{Z}) = \{ \chi \in \text{Hom}(G, \mathbb{R}) \setminus \{0\} \mid \text{the monoid } G_\chi \text{ is of type FP}_k \}$

Here, $G$ is of type $\text{FP}_k$ if there is a projective $\mathbb{Z}G$-resolution $P_\bullet \to \mathbb{Z}$, with $P_i$ finitely generated for all $i \leq k$. 
The BNSR invariants $\Sigma^q(G, \mathbb{Z})$ form a descending chain of open subsets of $\text{Hom}(G, \mathbb{R}) \setminus \{0\}$.

$\Sigma^k(G, \mathbb{Z}) \neq \emptyset \implies G$ is of type FP$_k$.

$\Sigma^1(G, \mathbb{Z}) = \Sigma^1(G)$.

The $\Sigma$-invariants control the finiteness properties of normal subgroups $N \triangleleft G$ with $G/N$ is abelian:

$$N \text{ is of type FP}_k \iff S(G, N) \subseteq \Sigma^k(G, \mathbb{Z})$$

where $S(G, N) = \{ \chi \in \text{Hom}(G, \mathbb{R}) \setminus \{0\} \mid \chi(N) = 0 \}$.

In particular:

$$\ker(\chi : G \to \mathbb{Z}) \text{ is f.g.} \iff \{ \pm \chi \} \subseteq \Sigma^1(G)$$
Let $X$ be a connected CW-complex with finite 1-skeleton, $G = \pi_1(X)$.

**Definition**

The *Novikov-Sikorav completion* of $\mathbb{Z}G$:

$$\hat{\mathbb{Z}}G_\chi = \left\{ \lambda \in \mathbb{Z}^G \mid \{ g \in \text{supp} \lambda \mid \chi(g) < c \} \text{ is finite, } \forall c \in \mathbb{R} \right\}$$

$\hat{\mathbb{Z}}G_\chi$ is a ring, contains $\mathbb{Z}G$ as a subring $\implies \hat{\mathbb{Z}}G_\chi$ is a $\mathbb{Z}G$-module.

**Definition**

$$\Sigma^q(X, \mathbb{Z}) = \{ \chi \in \text{Hom}(G, \mathbb{R}) \setminus \{0\} \mid H_i(X, \hat{\mathbb{Z}}G_{-\chi}) = 0, \forall i \leq q \}$$

Bieri: $G$ of type $\text{FP}_k$ $\implies \Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G, 1), \mathbb{Z}), \forall q \leq k$. 
Exponential tangent cone upper bound

**Theorem (Papadima–S.)**

*If X has finite k-skeleton, then, for every q ≤ k,*

\[
\Sigma^q(X, \mathbb{Z}) \subseteq \left( \tau_1^\mathbb{R} \left( \bigcup_{i \leq q} V_i^1(X) \right) \right)^c.
\]

(*)

Thus: Each Σ-invariant is contained in the complement of a union of rationally defined subspaces. Bound is sharp:

**Example**

Let G be a finitely generated nilpotent group. Then

\[
\Sigma^q(G, \mathbb{Z}) = \text{Hom}(G, \mathbb{R}) \setminus \{0\}, \quad V_1^q(G) = \{1\}, \quad \forall q
\]

and so (*) holds as an equality.
Resonance varieties

Let $X$ be a connected CW-complex with finite $k$-skeleton ($k \geq 1$). Let $A = H^*(X, \mathbb{C})$. Then: $a \in A^1 \Rightarrow a^2 = 0$. Thus, get cochain complex

$$ (A, \cdot a): A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots $$

Definition

The resonance varieties of $X$ are the algebraic sets

$$ \mathcal{R}^i_d(X) = \{ a \in A^1 \mid \dim_{\mathbb{k}} H^i(A, a) \geq d \}, $$

defined for all integers $0 \leq i \leq k$ and $d > 0$.

- $\mathcal{R}^i_d$ are homogeneous subvarieties of $A^1 = H^1(X, \mathbb{C})$
- $\mathcal{R}^i_1 \supseteq \mathcal{R}^i_2 \supseteq \cdots \supseteq \mathcal{R}^i_{b_i+1} = \emptyset$, where $b_i = b_i(X)$.
- $\mathcal{R}_d(X) = \mathcal{R}^1_d(X)$ depends only on $G = \pi_1(X)$. Write as $\mathcal{R}_d(G)$. 
Equivalent definition:

\[ R_d(X) = \left\{ a \in H^1(X, \mathbb{C}) \mid \exists \text{ subspace } W \subset H^1(X, \mathbb{C}) \text{ such that } \dim W = d + 1 \text{ and } a \cup W = 0 \right\} \]

In particular, \( 0 \neq a \in H^1(X, \mathbb{C}) \) belongs to \( R_1(X) \) iff there is \( b \in H^1(X) \) not proportional to \( a \), such that \( a \cup b = 0 \) in \( H^2(X) \).

**Example**

- \( R_1(T^n) = \{0\} \), for all \( n > 0 \).
- \( R_1(\vee^n S^1) = \mathbb{C}^n \), for all \( n > 1 \).
- \( R_1(\Sigma g) = \mathbb{C}^{2g} \), for all \( g > 1 \).

**Theorem (Libgober 2002)**

\[ TC_1(V_d^i(X)) \subseteq R_d^i(X) \]

Equality does not hold in general (Matei–S. 2002)
Formality

Definition

1. A group $G$ is 1-\textit{formal} if its Malcev Lie algebra, $m_{G} = \text{Prim}(\widehat{\mathbb{Q}G})$, is quadratic.

2. A space $X$ is \textit{formal} if its minimal model is quasi-isomorphic to $(H^{*}(X, \mathbb{Q}), 0)$.

- $X$ formal $\implies \pi_{1}(X)$ is 1-formal.
- $X_{1}, X_{2}$ formal $\implies X_{1} \times X_{2}$ and $X_{1} \vee X_{2}$ are formal
- $G_{1}, G_{2}$ 1-formal $\implies G_{1} \times G_{2}$ and $G_{1} \ast G_{2}$ are 1-formal
- $M_{1}, M_{2}$ formal, closed $n$-manifolds $\implies M_{1} \# M_{2}$ formal
Theorem (Dimca–Papadima–S.)

If $G$ is 1-formal, then $\exp : (\mathcal{R}_d(G), 0) \xrightarrow{\simeq} (\mathcal{V}_d(G, \mathbb{C}), 1)$. Hence

$$\tau_1(\mathcal{V}_d(G)) = TC_1(\mathcal{V}_d(G)) = \mathcal{R}_d(G)$$

In particular, $\mathcal{R}_d(G)$ is a union of rationally defined linear subspaces in $H^1(G, \mathbb{C}) = \text{Hom}(G, \mathbb{C})$.

Example

Let $G = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2], [x_1, x_4][x_2^{-2}, x_3], [x_1^{-1}, x_3][x_2, x_4] \rangle$. Then

$$\mathcal{R}_1(G) = \{ x \in \mathbb{C}^4 \mid x_1^2 - 2x_2^2 = 0 \}$$

splits into subspaces over $\mathbb{R}$ but not over $\mathbb{Q}$. Thus, $G$ is not 1-formal.
Example

- $X = F(\Sigma_g, n)$: the configuration space of $n$ labeled points of a Riemann surface of genus $g$ (a smooth, quasi-projective variety).
- $\pi_1(X) = P_{g,n}$: the pure braid group on $n$ strings on $\Sigma_g$.

Using computation of $H^*(F(\Sigma_g, n), \mathbb{C})$ by Totaro (1996), get

$$R_1(P_{1,n}) = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \left| \begin{array}{l} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 0, \\ x_i y_j - x_j y_i = 0, \text{ for } 1 \leq i < j < n \end{array} \right. \right\}$$

For $n \geq 3$, this is an irreducible, non-linear variety (a rational normal scroll). Hence, $P_{1,n}$ is not 1-formal.
Resonance upper bound

Corollary

Suppose \( \exp: (\mathcal{R}_1^{i}(X), 0) \overset{\simeq}{\longrightarrow} (\mathcal{V}_1^{i}(X), 1), \) for \( i \leq q. \) Then:

\[
\Sigma^{q}(X, \mathbb{Z}) \subseteq \left( \bigcup_{i \leq q} \mathcal{R}_1^{i}(X, \mathbb{R}) \right)^{\mathbb{C}}.
\]

Corollary

Suppose \( G \) is a 1-formal group. Then \( \Sigma^{1}(G) \subseteq \mathcal{R}_1(G, \mathbb{R})^{\mathbb{C}}. \)

In particular, if \( \mathcal{R}_1(G, \mathbb{R}) = H^{1}(G, \mathbb{R}) \), then \( \Sigma^{1}(G) = \emptyset. \)

Example

The above inclusion may be strict: Let \( G = \langle a, b \mid aba^{-1} = b^{2} \rangle. \) Then \( G \) is 1-formal, \( \Sigma^{1}(G) = (-\infty, 0) \), yet \( \mathcal{R}_1(G, \mathbb{R}) = \{0\}. \)
Kähler manifolds and Kähler groups

**Definition**
A compact, connected, complex manifold $M$ is called a *Kähler manifold* if $M$ admits a Hermitian metric $h$ for which the imaginary part $\omega = \Im(h)$ is a closed 2-form.

Examples: Riemann surfaces, $\mathbb{CP}^n$, and, more generally, smooth, complex projective varieties.

**Definition**
A group $G$ is a *Kähler group* if $G = \pi_1(M)$, for some compact Kähler manifold $M$.

$G$ is *projective* if $M$ is actually a smooth projective variety.

- $G$ finite $\implies G$ is a projective group (Serre 1958).
- $G_1, G_2$ Kähler groups $\implies G_1 \times G_2$ is a Kähler group
- $G$ Kähler group, $H \triangleleft G$ finite-index subgroup $\implies H$ is a Kähler gp
Problem (J.-P. Serre 1958)

Which finitely presented groups are Kähler (or projective) groups?

The Kähler condition puts strong restrictions on $M$:

1. $H^*(M, \mathbb{Z})$ admits a Hodge structure
2. Hence, the odd Betti numbers of $M$ are even
3. $M$ is formal (Deligne–Griffiths–Morgan–Sullivan 1975)

The Kähler condition also puts strong restrictions on $G = \pi_1(M)$:

1. $b_1(G)$ is even
2. $G$ is 1-formal
3. $G$ cannot split non-trivially as a free product (Gromov 1989)
Quasi-Kähler manifolds

**Definition**

A manifold $X$ is called *quasi-Kähler* if $X = X \setminus D$, where $X$ is Kähler and $D$ is a divisor with normal crossings.

- Smooth quasi-projective varieties (e.g., complements of hypersurfaces in $\mathbb{CP}^n$) are quasi-Kähler manifolds.
- A finitely-presented group $G$ is called a *quasi-Kähler group* if there is a quasi-Kähler manifold $X$ with $G = \pi_1(X)$.
- $X = \mathbb{CP}^n \setminus \{\text{hyperplane arrangement}\} \Rightarrow X$ is formal \((\text{Brieskorn 1973})\)
- $X$ quasi-projective, $W_1(H^1(X, \mathbb{C})) = 0 \Rightarrow \pi_1(X)$ is 1-formal \((\text{Morgan 1978})\)
- $X = \mathbb{CP}^n \setminus \{\text{hypersurface}\} \Rightarrow \pi_1(X)$ is 1-formal \((\text{Kohno 1983})\)
Theorem (Arapura 1997)

Let $G$ be a quasi-Kähler group. Then

$$\mathcal{V}_1(G) = \bigcup_{\alpha} \rho_\alpha T_\alpha$$

where $T_\alpha$ is an algebraic subtorus of $\text{Hom}(G, \mathbb{C}^\times)$ and $\rho_\alpha$ is a finite-order character.

Theorem (Dimca–Papadima–S.)

Let $G$ be a quasi-Kähler group, and $\Delta_G$ its Alexander polynomial.

- If $b_1(G) \neq 2$, then the Newton polytope of $\Delta_G$ is a line segment.
- If $G$ is actually a Kähler group, then $\Delta_G \equiv \text{const.}$
Resonance varieties of quasi-Kähler manifolds

Theorem (Dimca–Papadima–S.)

Let \( X \) be a quasi-Kähler manifold, and \( G = \pi_1(X) \). Let \( \{L_\alpha\}_\alpha \) be the non-zero irred components of \( \mathcal{R}_1(G) \). If \( G \) is 1-formal, then

1. Each \( L_\alpha \) is a linear subspace of \( H^1(G, \mathbb{C}) \).
2. Each \( L_\alpha \) is \( p \)-isotropic (i.e., restriction of \( \bigcup_G \) to \( L_\alpha \) has rank \( p \)), with \( \dim L_\alpha \geq 2p + 2 \), for some \( p = p(\alpha) \in \{0, 1\} \).
3. If \( \alpha \neq \beta \), then \( L_\alpha \cap L_\beta = \{0\} \).
4. \( \mathcal{R}_d(G) = \{0\} \cup \bigcup_{\alpha: \dim L_\alpha > d+p(\alpha)} L_\alpha \).

Furthermore,

4. If \( X \) is compact, then \( G \) is 1-formal, and each \( L_\alpha \) is 1-isotropic.
5. If \( W_1(H^1(X, \mathbb{C})) = 0 \), then \( G \) is 1-formal, and each \( L_\alpha \) is 0-isotropic.
**Σ-invariants**

Let $X$ be a quasi-Kähler manifold, $G = \pi_1(X)$.

**Theorem (Papadima–S.)**

1. $\Sigma^1(G) \subseteq TC^R_1(V^1_1(G))^c$.

2. If $X$ is Kähler, or $W_1(H^1(X, \mathbb{C})) = 0$, then $\mathcal{R}_1(G, \mathbb{R})$ is a finite union of rationally defined linear subspaces, and $\Sigma^1(G) \subseteq \mathcal{R}_1(G, \mathbb{R})^c$.

**Example**

Assumption from (2) is necessary. E.g., let $X$ be the complex Heisenberg manifold: bundle $\mathbb{C}^\times \to X \to (\mathbb{C}^\times)^2$ with $e = 1$. Then:

1. $X$ is a smooth quasi-projective variety;
2. $G = \pi_1(X)$ is nilpotent (and not 1-formal);
3. $\Sigma^1(G) = \mathbb{R}^2 \setminus \{0\}$ and $\mathcal{R}_1(G, \mathbb{R}) = \mathbb{R}^2$.

Thus, $\Sigma^1(G) \not\subseteq \mathcal{R}_1(G, \mathbb{R})^c$. 
Toric complexes and right-angled Artin groups

- $L$ simplicial complex on $n$ vertices $\leadsto$ toric complex $T_L$
  (subcomplex of $T^n$ obtained by deleting the cells corresponding to the missing simplices of $L$).
- $\pi_1(T_L)$ is the right-angled Artin group associated to graph $\Gamma = L^{(1)}$:
  $$G_{\Gamma} = \langle v \in V(\Gamma) \mid vw = wv \text{ if } \{v, w\} \in E(\Gamma) \rangle.$$ 
- $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$, where $\Delta_\Gamma$ is the flag complex of $\Gamma$.
- $H^*(T_L, \mathbb{Z})$ is the exterior Stanley-Reisner ring of $L$, with generators the duals $v^*$, and relations the monomials corresponding to the missing simplices of $L$.
- $T_L$ is formal, and so $G_\Gamma$ is 1-formal.

Example

- $\Gamma = \overline{K}_n \Rightarrow G_\Gamma = F_n$
- $\Gamma = K_n \Rightarrow G_\Gamma = \mathbb{Z}^n$
- $\Gamma = \Gamma' \coprod \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} \ast G_{\Gamma''}$
- $\Gamma = \Gamma' \ast \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} \times G_{\Gamma''}$
Identify $H^1(T_L, \mathbb{C}) = \mathbb{C}^V$, the $\mathbb{C}$-vector space with basis $\{v \mid v \in V\}$.

**Theorem (Papadima–S.)**

$$\mathcal{R}_d^i(T_L) = \bigcup_{W \subseteq V} \mathbb{C}^W,$$

$$\sum_{\sigma \in L \setminus W} \dim_{\mathbb{C}} \tilde{H}_{i-1-|\sigma|}(\text{lk}_{L_W}(\sigma), \mathbb{C}) \geq d$$

where $L_W$ is the subcomplex induced by $L$ on $W$, and $\text{lk}_K(\sigma)$ is the link of a simplex $\sigma$ in a subcomplex $K \subseteq L$.

In particular:

$$\mathcal{R}_1(G_\Gamma) = \bigcup_{W \subseteq V} \mathbb{C}^W.$$

$\Gamma_W$ disconnected
Similar formula holds for $\nu_d^i(T_L)$, with $C^W$ replaced by $(\mathbb{C}^\times)^W$. Hence:

$$\exp: (R_d^i(T_L), 0) \xrightarrow{\sim} (\nu_d^i(T_L), 1).$$

Using (1) resonance upper bound, and (2) computation of $\Sigma^k(G_\Gamma, \mathbb{Z})$ by Meier, Meinert, VanWyk (1998), we get:

### Corollary (Papadima-S.)

$$\Sigma^k(T_L, \mathbb{Z}) \subseteq \left( \bigcup_{i \leq k} R_1^i(T_L, \mathbb{R}) \right)^c$$

$$\Sigma^k(G_\Gamma, \mathbb{Z}) = \left( \bigcup_{i \leq k} R_1^i(T_\Delta_\Gamma, \mathbb{R}) \right)^c$$
Example

\[ \Gamma = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} \]

Maximal disconnected subgraphs: \( \Gamma \{134\} \) and \( \Gamma \{124\} \). Thus:

\[ R_1(G_\Gamma) = C\{134\} \cup C\{124\}. \]

Note that:

\[ C\{134\} \cap C\{124\} = C\{14\} \neq \{0\} \]

Since \( G_\Gamma \) is 1-formal \( \Rightarrow G_\Gamma \) is not a quasi-Kähler group.

Theorem (Dimca–Papadima–S.)

The following are equivalent:

1. \( G_\Gamma \) is a quasi-Kähler group
2. \( \Gamma = K_{n_1},...,n_r := \overline{K}_{n_1} \ast \cdots \ast \overline{K}_{n_r} \)
3. \( G_\Gamma = F_{n_1} \times \cdots \times F_{n_r} \)

1. \( G_\Gamma \) is a Kähler group
2. \( \Gamma = K_{2r} \)
3. \( G_\Gamma = \mathbb{Z}^{2r} \)
Bestvina–Brady groups

\[ N_\Gamma = \ker(\nu: G_\Gamma \to \mathbb{Z}), \text{ where } \nu(\nu) = 1, \text{ for all } \nu \in V(\Gamma). \]

**Theorem (Dimca–Papadima–S.)**

The following are equivalent:

1. \( N_\Gamma \) is a quasi-Kähler group
2. \( \Gamma \) is either a tree, or \( \Gamma = K_{n_1}, \ldots, n_r \), with some \( n_i = 1 \), or all \( n_i \geq 2 \) and \( r \geq 3 \).
3. \( N_\Gamma \) is a Kähler group
4. \( \Gamma = K_{2r+1} \)
5. \( N_\Gamma = \mathbb{Z}^{2r} \)

**Example (answers a question of J. Kollár)**

\( \Gamma = K_{2,2,2} \rightsquigarrow G_\Gamma = F_2 \times F_2 \times F_2 \rightsquigarrow N_\Gamma = \text{the Stallings group} \)

\( N_\Gamma \) is finitely presented, but rank \( H_3(N_\Gamma, \mathbb{Z}) = \infty \), so \( N_\Gamma \) not FP\(_3\).

Also, \( N_\Gamma = \pi_1(\mathbb{C}^2 \setminus \{\text{an arrangement of 5 lines}\}) \).

Thus, \( N_\Gamma \) is a quasi-projective group which is not commensurable (even up to finite kernels) to any group \( \pi \) having a finite \( K(\pi, 1) \).
3-manifolds

Question (Donaldson–Goldman 1989, Reznikov 1993)
Which 3-manifold groups are Kähler groups?


Theorem (Dimca–S.)
Let $G$ be the fundamental group of a closed 3-manifold. Then $G$ is a Kähler group $\iff G$ is a finite subgroup of $\text{O}(4)$, acting freely on $S^3$.

Idea of proof: compare the resonance varieties of (orientable) 3-manifolds to those of Kähler manifolds.
Proposition

Let $M$ be a closed, orientable 3-manifold. Then:

1. $H^1(M, \mathbb{C})$ is not 1-isotropic.
2. If $b_1(M)$ is even, then $\mathcal{R}_1(M) = H^1(M, \mathbb{C})$.

On the other hand, it follows from [Dimca–Papadima–S.] that:

Proposition

Let $M$ be a compact Kähler manifold with $b_1(M) \neq 0$. If $\mathcal{R}_1(M) = H^1(M, \mathbb{C})$, then $H^1(M, \mathbb{C})$ is 1-isotropic.

But $G = \pi_1(M)$, with $M$ Kähler $\Rightarrow b_1(G)$ even. Thus, if $G$ is both a 3-mfd group and a Kähler group $\Rightarrow b_1(G) = 0$. Using work of Fujiwara (1999) and Reznikov (2002) on Kazhdan’s property (T), as well as Perelman (2003) $\Rightarrow G$ finite subgroup of $O(4)$.
Question

Which 3-manifold groups are quasi-Kähler groups?

Theorem (Dimca–Papadima–S.)

Let $G$ be the fundamental group of a closed, orientable 3-manifold. Assume $G$ is 1-formal. Then the following are equivalent:

1. $m(G) \cong m(\pi_1(X))$, for some quasi-Kähler manifold $X$.
2. $m(G) \cong m(\pi_1(M))$, where $M$ is either $S^3$, $\#^n S^1 \times S^2$, or $S^1 \times \Sigma_g$. 
BNS invariant and Thurston norm

Let $M$ be a compact, connected 3-manifold, with $G = \pi_1(M)$.

**Theorem (Bieri–Neumann–Strebel 1987)**

- $\Sigma^1(G) = \bigcup F$ fibered face of Thurston norm unit ball $\mathbb{R}_+ \cdot F$.
- $\Sigma^1(G) = -\Sigma^1(G)$.
- $M$ fibers over $S^1 \iff \Sigma^1(G) \neq \emptyset$.

Using (1) upper bound $\Sigma^1(G) \subseteq R_1(G, \mathbb{R})^c$ for 1-formal groups, and (2) description of $R_1(M^3)$ from above, we get:

**Corollary (Papadima–S.)**

Let $M$ be a closed, orientable 3-manifold. If $b_1(M)$ is even, and $G = \pi_1(M)$ is 1-formal, then $M$ does not fiber over the circle.


