Northeastern University
Department of Mathematics

Qualifying Exam in Topology

Winter 2004

Do six of the following seven problems. Give proofs or justifications for each statement you make. Draw pictures when needed. Be as clear and concise as possible. Show all your work.

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1. Let X be a topological space, and U an open subset. Show that the set $A = \overline{U} \setminus U$ is nowhere dense, i.e., $\operatorname{int}(\overline{A}) = \emptyset$.

2. Prove or disprove the following:
   (a) The continuous image of a locally connected space is locally connected.
   (b) The quotient of a locally connected space is locally connected.

3. Let X be a connected, locally path-connected space. Suppose $\pi_1(X)$ is finite. Show that every continuous map $f\colon X \to S^1$ is homotopic to a constant map.

4. Let X be the space obtained from a solid hexagon by identifying opposite sides via parallel translation. Compute $\pi_1(X)$ and $H_*(X)$.

5. Let $X = S^1 \vee S^1$ be the wedge of two circles. Find all the connected, 3-fold covering spaces of X, up to equivalence of covers. In each case, draw a picture, and indicate whether the cover is normal or not.

6. Let T^2 be the 2-torus, and K the Klein bottle.
   (a) Define a two-fold covering map $p\colon T^2 \to K$.
   (b) Pick base-points $e \in T^2$ and $b \in K$ with $p(e) = b$. Determine the induced homomorphism $p_*\colon \pi_1(T^2, e) \to \pi_1(K, b)$.
   (c) Is p_* injective? If not, what is $\ker(p_*)$?
   (d) Is p_* surjective? If not, what is $\operatorname{coker}(p_*)$?
   (e) Determine the induced homomorphism $p_*\colon H_*(T^2; \mathbb{Z}) \to H_*(K; \mathbb{Z})$.

7. Let $X = \mathbb{RP}^2 \times \mathbb{RP}^2$ be the product of two copies of the projective plane.
   (a) Compute $\pi_1(X)$.
   (b) Find a CW-decomposition of X.
   (c) Determine the chain complex $(C_\bullet(X), d)$ associated to that cell decomposition.
   (d) Compute the homology groups $H_*(X)$.