Northeastern University
Department of Mathematics

Qualifying Exam in Topology

January 2008

Do the following six problems. Give proofs or justifications for each statement you make. Draw pictures when needed. Be as **clear** *and* **concise** *as possible. Show all your work.*

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1. Let $f, g\colon X \to Y$ be two continuous maps, from a topological space X to a Hausdorff space Y. Suppose $f = g$ on a subset $A \subset X$ which is dense in X. Show that $f = g$ on all of X.

2. Let (X, d) be a metric space, and let $f\colon X \to X$ be a continuous function which has no fixed points.
   (a) If X is compact, show that there is a real number $\epsilon > 0$ such that $d(x, f(x)) > \epsilon$, for all $x \in X$.
   (b) Show that the conclusion in (a) is false if X is not assumed to be compact.

3. Let X be a topological space.
   (a) Show that, if X is connected and locally path-connected, then X is path-connected.
   (b) Show that, if X is locally path-connected, then all the path-components of X are both open and closed.
   (c) If all the path-components of X are open, does it follow that X is locally path-connected?

4. Let X be the space obtained by attaching two disks, D_1 and D_2, to the circle S^1, where the first disk is attached via the map $f_1\colon \partial D_1 = S^1 \to S^1$, $f_1(z) = z^2$, and the second disk is attached via the map $f_2\colon \partial D_2 = S^1 \to S^1$, $f_2(z) = z^5$.
   (a) Use the Seifert-van Kampen theorem to compute the fundamental group $\pi_1(X, x_0)$.
   (b) Compute the homology groups $H_i(X, \mathbb{Z})$, for all $i \geq 0$.

5. Let T^2 be the 2-dimensional torus.
   (a) Identify (up to homeomorphism) all the path-connected spaces E that appear as the total space of a covering map $p\colon E \to T^2$. Which one of those is the universal cover?
   (b) Prove, or give a counterexample to the following assertion: Every continuous map $S^1 \to T^2$ is null-homotopic.
   (c) Prove, or give a counterexample to the following assertion: Every continuous map $S^2 \to T^2$ is null-homotopic.

6. Let $B = S^1 \vee S^1$ be the wedge of two circles. Find at least 4 non-equivalent 3-fold covering spaces $p\colon E \to B$, with E path-connected. In each case:
   (a) Draw a picture of the cover, clearly indicating how the projection map p works.
   (b) Identify $\pi_1(E, e)$ and $\pi_1(B, b)$, and compute the induced homomorphism, $p_\sharp\colon \pi_1(E, e) \to \pi_1(B, b)$, for some conveniently chosen basepoints e and b with $p(e) = b$.
   (c) Indicate whether the cover is regular or not.