Northeastern University
Department of Mathematics

Qualifying Exam in Topology

September 2009

Do the following five problems. Give proofs or justifications for each statement you make. Draw pictures when needed. Be as **clear** *and* **concise** *as possible. Show all your work.*

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1. Let $f\colon X \to Y$ be a continuous, surjective map from a space X to a connected space Y. Assume $f^{-1}(y)$ is connected, for each $y \in Y$.
   (a) Show that if f is a quotient map, then X is connected.
   (b) Give an example to show that if f is not a quotient map, then X need not be connected.

2. Let $f\colon X \to Y$ be a continuous map from a space X to a Hausdorff space Y. Let C be a closed subspace of Y, and let U be an open neighborhood of $f^{-1}(C)$ in X.
   (a) Show that if X is compact then there is an open neighborhood V of C in Y such that $f^{-1}(V)$ is contained in U.
   (b) Give an example to show that if X is not compact, then there need not be such a neighborhood V.

3. For an integer $n \geq 1$, let S^n be the n-sphere, $\mathbb{RP}^n$ the n-dimensional projective space, and T^n the n-dimensional torus.
   (a) For which values of n does there exist a continuous map $S^n \to S^1$ which is not homotopic to a constant?
   (b) For which values of n does there exist a continuous map $\mathbb{RP}^n \to S^1$ which is not homotopic to a constant?
   (c) For which values of n does there exist a continuous map $T^n \to S^1$ which is not homotopic to a constant?

4. Let A be a 2-holed torus (i.e., a compact, connected, orientable surface of genus 2) with an open 2-disk removed, and let B be another copy of the 2-disk.
   Let $f\colon \partial B \to \partial A$ be the map winding three times (upon identifying ∂A and ∂B with the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, the map f is given by $f(z) = z^3$).
   Finally, let $X = A \cup_f B$ be the space obtained from A by adjoining the 2-cell B along the attaching map f.
   (a) Compute the fundamental group of X.
   (b) Compute all the (integral) homology groups of X.

5. Let $B = S^1 \vee S^1$ be the wedge of two circles. Find at least 4 non-equivalent 3-fold covering spaces $p\colon E \to B$, with E path-connected. In each case:
   (a) Draw a picture of the cover, clearly indicating how the projection map p works.
   (b) Compute the induced homomorphism on fundamental groups, $p_\sharp\colon \pi_1(E, e) \to \pi_1(B, b)$, for some conveniently chosen basepoints e and b with $p(e) = b$.
   (c) Compute the induced homomorphism on first homology groups, $p_*\colon H_1(E, \mathbb{Z}) \to H_1(B, \mathbb{Z})$.
   (d) Indicate whether the cover is regular or not.