Northeastern University
Department of Mathematics

Qualifying Exam in Topology

April 2010

Do the following six problems. Give proofs or justifications for each statement you make. Draw pictures when needed. Be as **clear** *and* **concise** *as possible. Show all your work.*

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1. Let X be a Hausdorff space, and let A be a subspace of X. Suppose the inclusion map, $i\colon A \to X$, admits a retraction, i.e., suppose there is a continuous map $r\colon X \to A$ such that $r(i(a)) = a$, for every $a \in A$. Show that A is a closed subset of X.

2. Let $p\colon X \to Y$ be a quotient map. Suppose Y is connected, and, for each $y \in Y$, the subspace $p^{-1}(\{y\})$ is connected. Show that X is connected.

3. Let $Y = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \neq z_2\}$. Let $X = Y/\sigma$ be the quotient space of Y by the involution σ permuting the coordinates. Let $p\colon Y \to X$ be the projection map.
   (a) Find the fundamental group of Y, based at a point $y_0 \in Y$.
   (b) Find the fundamental group of X, based at $x_0 = p(y_0)$.
   (c) Determine the induced homomorphim $p_\sharp\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)$.

4. Let A and B be subsets of the sphere S^n, $n \geq 2$. Show:
   (a) If A and B are closed, disjoint, and neither separates S^n, then $A \cup B$ does not separate S^n.
   (b) If A and B are connected, open, and $A \cup B = S^n$, then $A \cap B$ is connected.

5. Let $B = S^1 \vee S^1$ be the wedge of two circles, and choose the wedge point b_0 as basepoint.
   (a) Construct a two-fold covering map $p\colon E \to B$, with E connected.
   (b) Find a subgroup H of $\pi_1(B, b_0)$ corresponding to p. Show that H is a normal subgroup. Describe the group of deck transformations of p.
   (c) Pick $e_0 \in p^{-1}(b_0)$, and determine the induced homomorphism $p_\sharp\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$.
   (d) Determine the induced homomorphism $p_*\colon H_1(E; \mathbb{Z}) \to H_1(B; \mathbb{Z})$.

6. Let $X = \mathbb{RP}^2 \times K$ be the product of the real projective plane with the Klein bottle.
   (a) Find a CW-decomposition of X.
   (b) Determine the chain complex $(C_\bullet(X), \partial)$ associated to this cell decomposition.
   (c) Use this chain complex to compute the homology groups $H_*(X, \mathbb{Z})$.