Qualifying Exam in Geometry

(1.5 hours)

Spring 1996

1. If $\omega = f dx$ is a 1-form on $[0,1]$, with $f(0) = f(1)$, show that there is a unique number λ such that

$$\omega = \lambda dx + dg$$

for some function g with $g(0) = g(1)$.

2. i) Prove that $\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ is not of the form e^A, for any $A \in \mathfrak{gl}(2, \mathbb{R})$.

ii) Find matrices $A, B \in \mathfrak{gl}(2, \mathbb{C})$ such that $e^{A+B} \neq e^A e^B$.

3. For each of the following spheres: S^1, S^2, and S^3, show that either

a) There is no nowhere-vanishing vector field, or

b) Exhibit a nowhere-vanishing vector field.

4. i) Let $\xi : E \xrightarrow{\pi} B$ be a C^∞ vector bundle. What does it mean that ξ is orientable?

ii) Let M be a C^∞ manifold. What does it mean that M is orientable?

iii) Show that there is an exact sequence of vector bundles

$$0 \to \pi^*(\xi) \to TE \to \pi^*(TB) \to 0.$$

iv) If $0 \to E_1 \to E_2 \to E_3 \to 0$ is exact, then each bundle E_i is orientable if the other two are.

v) Show that $T(TM)$ is always orientable.

vi) Assume the bundle $\xi : E \xrightarrow{\pi} B$ is not orientable. Show that the manifold E is not orientable.