

# PURE VIRTUAL BRAIDS, RESONANCE, AND FORMALITY

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**ABSTRACT.** We investigate the resonance varieties, lower central series ranks, and Chen ranks of the pure virtual braid groups and their upper-triangular subgroups. As an application, we give a complete answer to the 1-formality question for this class of groups. In the process, we explore various connections between the Alexander-type invariants of a finitely generated group and several of the graded Lie algebras associated to it, and discuss possible extensions of the resonance-Chen ranks formula in this context.

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## 1. INTRODUCTION

**1.1. Background.** Virtual knot theory, as introduced by Kauffman in [22], is an extension of classical knot theory. This new theory studies embeddings of knots in thickened surfaces of arbitrary genus, while the classical theory studies the embeddings of circles in thickened spheres. Another motivation comes from the representation of knots by Gauss diagrams. In [20], Goussarov, Polyak, and Viro showed that the usual knot theory embeds into virtual knot theory, by realizing any Gauss diagram by a virtual knot. Many knot invariants,

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such as knot groups, the bracket polynomial, and finite-type Vassiliev invariants can be extended to invariants of virtual knots, see [22, 20].

The virtual braid groups  $vB_n$  were introduced in [22] and further studied in [3, 6, 23, 4, 21]. As shown by Kamada in [21], any virtual link can be constructed as the closure of a virtual braid, which is unique up to certain Reidemeister-type moves. In this paper, we will be mostly interested in the kernel of the canonical epimorphism  $vB_n \rightarrow S_n$ , called the *pure virtual braid group*,  $vP_n$ , and a certain subgroup of this group,  $vP_n^+$ , which we call the *upper pure virtual braid group*.

In [8], Bartholdi, Enriquez, Etingof, and Rains independently defined and studied the group  $vP_n$ , which they called the  $n$ -th quasitriangular groups  $Q\text{Tr}_n$ , as a group-theoretic version of the set of solutions to the Yang–Baxter equations. Their work was developed in a deep way by P. Lee in [27]. The authors of [8, 27] construct a classifying space for  $vP_n$  with finitely many cells, and find a presentation for the cohomology algebra of  $vP_n$ , which they show is a Koszul algebra. They also obtain parallel results for a quotient group of  $vP_n$ , called the  $n$ -th triangular group, which has the same generators as  $vP_n$ , and one more set of relations. It is readily seen that the triangular group  $\text{Tr}_n$  is isomorphic to  $vP_n^+$ .

We refer to [43] for further context on the pure virtual braid groups and related subgroups of basis-conjugating automorphisms of free groups.

**1.2. Presentations and associated graded Lie algebras.** Bardakov gave in [4] a presentation for the pure virtual braid group  $vP_n$ , much simpler than the usual presentation of the pure braid group  $P_n$ . As shown in [8], there exists a monomorphism from  $P_n$  to  $vP_n$ . Moreover, there are split injections  $vP_n \rightarrow vP_{n+1}$ ,  $vP_n^+ \rightarrow vP_{n+1}^+$  and  $vP_n^+ \rightarrow vP_n$ .

The pure braid group  $P_n$  has center  $\mathbb{Z}$ , so there is a decomposition  $P_n \cong \overline{P}_n \times \mathbb{Z}$ . Using a decomposition of  $vP_3$  given by Bardakov, Mikhailov, Vershinin, and Wu in [7], we show that  $vP_3 \cong \overline{P}_4 * \mathbb{Z}$ . In this context, it is worth noting that the center of  $vP_n$  is trivial for  $n \geq 2$ , and the center of  $vP_n^+$  is trivial for  $n \geq 3$ , with one possible exception; see Dies and Nicas [15].

In a different vein, Labute [26] and Anick [1] defined the notion of a ‘mild’ presentation for a finitely presented group  $G$ . If the group  $G$  admits such a presentation, then a presentation for the (complex) associated graded Lie algebra  $\text{gr}(G)$  can be obtained from the classical Magnus expansion. In general, though, finding a presentation for this Lie algebra is an onerous task. In §5.5, we prove the following result.

**Proposition 1.1.** *The pure braid groups  $P_n$  and the pure virtual braid groups  $vP_n$  and  $vP_n^+$  admit mild presentations if and only  $n \leq 3$ .*

Nevertheless, explicit, quadratic presentations for the associated graded Lie algebras of the groups  $G_n = P_n$ ,  $vP_n$ , and  $vP_n^+$  were given in [24, 18] for  $P_n$ , and in [8, 27] for  $vP_n$  and  $vP_n^+$ . These computations also show that, for each of these groups, the universal enveloping algebra  $U(\text{gr}(G_n))$  is a Koszul algebra. Using Koszul duality and some combinatorial manipulations, we show in §5.4 that the LCS ranks of the groups  $G_n$

are given by

$$(1) \quad \phi_k(G_n) = \frac{1}{k} \sum_{d|k} \mu\left(\frac{k}{d}\right) \left[ \sum_{m_1+2m_2+\dots+nm_n=d} (-1)^{s_n} d(m!) \prod_{j=1}^n \frac{(b_{n,n-j})^{m_j}}{(m_j)!} \right],$$

where  $m_j$  are non-negative integers,  $s_n = \sum_{i=1}^{\lfloor n/2 \rfloor} m_{2i}$ ,  $m = \sum_{i=1}^n m_i - 1$ , and  $\mu$  is the Möbius function, while  $b_{n,j}$  are the (unsigned) Stirling numbers of the first kind (for  $G_n = P_n$ ), the Lah numbers (for  $G_n = vP_n$ ), or the Stirling numbers of the second kind (for  $G_n = vP_n^+$ ).

**1.3. Resonance and formality.** Consider now a group  $G$  admitting a finite-type classifying space  $K(G, 1)$ . The cohomology algebra  $H^*(G, \mathbb{C})$  may be turned into a family of cochain complexes parametrized by the affine space  $H^1(G, \mathbb{C})$ , from which one may define the *resonance varieties* of the group,  $\mathcal{R}_d^i(G)$ , as the loci where the cohomology of those cochain complexes jumps. After a brief review of these notions, we study in §4 the behavior of resonance under product and coproducts, obtaining formulas which generalize those from [35], see Propositions 4.2 and 4.3.

The formality and partial formality properties of spaces and groups are basic notions in homotopy theory, allowing one to describe the rational homotopy type of a simply-connected space, or the tower of nilpotent quotients of a group in terms of the rational cohomology ring of the object in question. More concretely, a finitely generated group  $G$  is said to be *1-formal* if its Malcev Lie algebra admits a quadratic presentation. Following [42], we will separate the 1-formality property into graded-formality and filtered-formality.

The work of Bartholdi et al. [8] and Lee [27] mentioned above shows that the pure virtual braid group  $vP_n$  and its subgroup  $vP_n^+$  are graded-formal, for all  $n$ . Furthermore, Bartholdi, Enriquez, Etingof, and Rains state that the groups  $vP_n$  and  $vP_n^+$  are not 1-formal for  $n \geq 4$ , and sketch a proof of this claim. One of the aims of this paper (indeed, the original motivation for this work) is to provide a detailed proof of this fact. Our main result, which is proved in §6, reads as follows.

**Theorem 1.2.** *The groups  $vP_n$  and  $vP_n^+$  are both 1-formal if  $n \leq 3$ , and they are both non-1-formal (and thus, not filtered formal) if  $n \geq 4$ .*

The 1-formality property of groups is preserved under split injections and (co)products, see [42, 17]. Consequently, the fact that we have split injections between the various pure virtual braid groups allows us to reduce the proof of Theorem 1.2 to verifying the 1-formality of  $vP_3$  and the non-1-formality of  $vP_4^+$ . To prove the first statement, we use the free product decomposition  $vP_3 \cong \mathbb{Z} * \overline{P}_4$ . For the second statement, we compute the resonance variety  $\mathcal{R}_1^1(vP_4^+)$ , and use the geometry of this variety, together with the Tangent Cone Theorem from [17] to reach the desired conclusion.

**1.4. Chen ranks and Alexander invariants.** Given a finitely generated group  $G$ , we let  $\theta_k(G) = \dim \text{gr}_k(G/G'')$  be the LCS ranks of the maximal metabelian quotient of  $G$ . These ranks were first studied by K.-T. Chen in [10], and are named after him. In §9, we

give the generating functions for the Chen ranks of the free groups  $F_n$  and the pure braid groups  $P_n$ , based on computations from [10, 11].

As shown by W. Massey in [30], the Chen ranks can be computed from the Alexander invariant,  $B(G) = G'/G''$ . More precisely, if we view this abelian group as a module over  $\mathbb{C}[H]$ , where  $H = G/G'$ , then filter it by powers of the augmentation ideal, and take the associated graded module,  $\text{gr}(B(G))$ , viewed as a module over the symmetric algebra  $S = \text{Sym}(H \otimes \mathbb{C})$ , we have that  $\theta_{k+2}(G) = \dim \text{gr}_k(B(G))$  for all  $k \geq 0$ .

In a similar fashion, we define the Chen ranks of a finitely generated, graded Lie algebra  $\mathfrak{g}$  to be  $\theta_k(\mathfrak{g}) = \dim(\mathfrak{g}/\mathfrak{g}'')_k$ , and the infinitesimal Alexander invariant of  $\mathfrak{g}$  to be the graded  $\text{Sym}(\mathfrak{g}_1)$ -module  $\mathfrak{B}(\mathfrak{g}) = \mathfrak{g}'/\mathfrak{g}''$ , after which we show that  $\theta_{k+2}(\mathfrak{g}) = \dim \mathfrak{B}(\mathfrak{g})_k$  for all  $k \geq 0$ .

Our next main result (a combination of Propositions 8.2, 8.4, and 8.5), relates the various Alexander-type invariants associated to a group, as follows.

**Theorem 1.3.** *Let  $G$  be a finitely generated group with abelianization  $H$ , and set  $S = \text{Sym}(H \otimes \mathbb{C})$ . There exists then surjective morphisms of graded  $S$ -modules,*

$$(2) \quad \mathfrak{B}(\mathfrak{h}(G)) \xrightarrow{\psi} \mathfrak{B}(\text{gr}(G)) \xrightarrow{\varphi} \text{gr}(B(G)).$$

Moreover, if  $G$  is graded-formal, then  $\psi$  is an isomorphism, and if  $G$  is filtered-formal, then  $\varphi$  is an isomorphism.

This result yields the following inequalities between the various types of Chen ranks associated to a finitely generated group  $G$ :

$$(3) \quad \theta_k(\mathfrak{h}(G)) \geq \theta_k(\text{gr}(G)) \geq \theta_k(G),$$

with the first inequality holding as equality if  $G$  is graded-formal, and the second inequality holding as equality if  $G$  is filtered-formal.

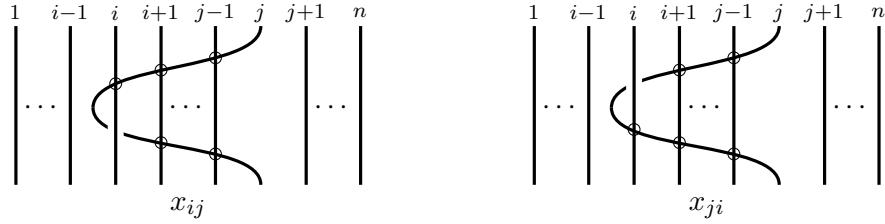
**1.5. The Chen ranks formula.** Now suppose  $G$  is a finitely presented, commutator-relators group. Then, as shown in [33], the infinitesimal Alexander invariant  $\mathfrak{B}(\mathfrak{h}(G))$  coincides with the ‘linearization’ of the Alexander invariant  $B(G)$ . Furthermore, as shown in [32], the resonance variety  $\mathcal{R}_1^1(G)$  coincides, away from the origin  $0 \in H^1(G; \mathbb{C})$ , with the support variety of the annihilator of  $\mathfrak{B}(\mathfrak{h}(G))$ .

Under certain conditions, the *Chen ranks formula* reveals a close relationship between the first resonance variety of  $G$  and the Chen ranks, namely,

$$(4) \quad \theta_k(G) = \sum_{m \geq 2} h_m(G) \cdot \theta_k(F_m)$$

for  $k \gg 1$ , where  $h_m(G)$  is the number of  $m$ -dimensional irreducible components of  $\mathcal{R}_1^1(G)$ . This formula was conjectured in [40] for arrangement groups and proved by Cohen and Schenck in [14] for 1-formal groups satisfying certain restrictions on their resonance components.

In §9, we analyze the Chen ranks formula in a wider setting, with a view towards comparing the Chen ranks and the resonance varieties of the pure virtual braid groups.

FIGURE 1. The virtual pure braids  $x_{ij}$  and  $x_{ji}$  for  $i < j$ .

We start by noting that formula (4) may hold even for non-1-formal groups, such as the fundamental groups of complements of suitably chosen arrangements of planes in  $\mathbb{R}^4$ .

Next, we look at the way the Chen ranks formula behaves well with respect to products and coproducts of groups. The conclusion may be summarized as follows.

**Proposition 1.4.** *If both  $G_1$  and  $G_2$  satisfy the Chen ranks formula (4), then  $G_1 \times G_2$  also satisfies the Chen ranks formula, but  $G_1 * G_2$  may not.*

Finally, we analyze the Chen ranks and the degree 1 resonance varieties of the groups  $vP_n$  and  $vP_n^+$ . In a preprint version of [7], Bardakov et al. compute some of the degree 1 and 2 resonance varieties of  $vP_3$ . Using the decomposition  $vP_3 \cong \mathbb{Z} * \overline{P}_4$ , we compute all the resonance varieties of this group. We also compute the ranks  $\theta_k(vP_3)$ , and show that formula (4) does not hold in this case, although  $vP_3$  satisfies all but one of the hypothesis of [14, Theorem A]. Likewise, we compute the Chen ranks and degree 1 resonance varieties of the groups  $vP_n^+$  for low values of  $n$ , and conclude that  $vP_4^+$  and  $vP_5^+$  also do not satisfy the Chen ranks formula.

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## 2. PURE VIRTUAL BRAID GROUPS

In this section, we look at the pure virtual braid groups and their upper-triangular subgroups from the point view of combinatorial group theory.

**2.1. Presentations.** Let  $P_n$  be the Artin pure braid group on  $n$  strings. As is well-known, the center of  $P_n$  ( $n \geq 2$ ) is infinite cyclic, and so we have a direct product decomposition of the form  $P_n \cong \overline{P}_n \times \mathbb{Z}$ . The first few groups in this series are easy to describe:  $P_1 = \{1\}$ ,  $P_2 = \mathbb{Z}$ , and  $P_3 \cong F_2 \times \mathbb{Z}$ , where  $F_n$  denotes the free group on  $n$  generators.

As shown in [4], the *pure virtual braid group*  $vP_n$  has presentation with generators  $x_{ij}$  with  $1 \leq i \neq j \leq n$  (see Figure 1 for a description of the corresponding virtual braids), and relations

$$(5) \quad \begin{aligned} x_{ij}x_{ik}x_{jk} &= x_{jk}x_{ik}x_{ij} && \text{for distinct } i, j, k, \\ x_{ij}x_{kl} &= x_{kl}x_{ij} && \text{for distinct } i, j, k, l. \end{aligned}$$

The *upper-triangular pure virtual braid group*,  $vP_n^+$ , is the subgroup of  $vP_n$  generated by those elements  $x_{ij}$  with  $1 \leq i < j \leq n$ . Its defining relations are

$$(6) \quad \begin{aligned} x_{ij}x_{ik}x_{jk} &= x_{jk}x_{ik}x_{ij} && \text{for } i < j < k, \\ x_{ij}x_{kl} &= x_{kl}x_{ij} && \text{for } i \neq j \neq k \neq l, i < j, \text{ and } k < l. \end{aligned}$$

Of course, both  $vP_1$  and  $vP_1^+$  are the trivial group. It is readily seen that  $vP_2^+ = \mathbb{Z}$ , while  $vP_3^+ \cong \mathbb{Z} * \mathbb{Z}^2$ . Likewise,  $vP_2$  is isomorphic to  $F_2$ .

**2.2. Split monomorphisms.** In [8], the group  $vP_n$  is called the *quasi-triangular group*, and is denoted by  $\text{QTr}_n$ , while the quotient group of  $\text{QTr}_n$  by the relations of the form  $x_{ij} = x_{ji}$  for  $i \neq j$  is called the *triangular group*, and is denoted by  $\text{Tr}_n$ .

**Lemma 2.1.** *The group  $\text{Tr}_n$  is isomorphic to  $vP_n^+$ .*

*Proof.* Let  $\phi: \text{Tr}_n \rightarrow vP_n^+$  be the homomorphism defined by  $\phi(x_{ij}) = x_{ij}$  for  $i < j$  and  $\phi(x_{ij}) = x_{ji}$  for  $i > j$ , and let  $\psi: vP_n^+ \rightarrow \text{Tr}_n$  be the homomorphism defined by  $\psi(x_{ij}) = x_{ij}$  for  $i < j$ . It is easy to show that  $\phi$  and  $\psi$  are well-defined homomorphisms, and  $\phi \circ \psi = \text{id}$  and  $\psi \circ \phi = \text{id}$ . Thus,  $\phi$  is an isomorphism.  $\square$

**Corollary 2.2.** *The inclusion  $j_n: vP_n^+ \hookrightarrow vP_n$  is a split monomorphism.*

*Proof.* The split surjection is defined by the composition of the quotient surjection  $vP_n \twoheadrightarrow \text{Tr}_n$  and the map  $\phi: \text{Tr}_n \rightarrow vP_n^+$  from Lemma 2.1.  $\square$

There are several other split monomorphisms between the aforementioned groups.

**Lemma 2.3.** *For each  $n \geq 2$ , there are split monomorphisms  $\iota_n: vP_n \rightarrow vP_{n+1}$  and  $\iota_n^+: vP_n^+ \rightarrow vP_{n+1}^+$ .*

*Proof.* The maps  $\iota_n$  and  $\iota_n^+$  are defined by sending the generators of  $vP_n$  to the generators of  $vP_{n+1}$  with the same indices. The split surjection  $\pi_n: vP_{n+1} \twoheadrightarrow vP_n$  sends  $x_{ij}$  to zero for  $i = n + 1$  or  $j = n + 1$ , and sends  $x_{ij}$  to  $x_{ij}$  otherwise. The split surjection  $\pi_n^+: vP_{n+1}^+ \twoheadrightarrow vP_n^+$  is defined similarly.  $\square$

As noted by Bardakov in [4, Lemma 6], the pure virtual braid group  $vP_n$  admits a semi-direct product decomposition of the form  $vP_n \cong F_{q(n)} \rtimes vP_{n-1}$ , where  $q(1) = 2$  and  $q(n)$  is infinite for  $n \geq 2$ . Furthermore, as shown in [8], there exists a monomorphism from  $P_n$  to  $vP_n$ .

**2.3. A free product decomposition for  $vP_3$ .** The pure virtual braid group  $vP_3$  is generated by  $x_{12}, x_{21}, x_{13}, x_{31}, x_{23}, x_{32}$ , subject to the relations

$$\begin{aligned} x_{12}x_{13}x_{23} &= x_{23}x_{13}x_{12}, & x_{21}x_{23}x_{13} &= x_{13}x_{23}x_{21}, & x_{13}x_{12}x_{32} &= x_{32}x_{12}x_{13}, \\ x_{31}x_{32}x_{12} &= x_{12}x_{32}x_{31}, & x_{23}x_{21}x_{31} &= x_{31}x_{21}x_{23}, & x_{32}x_{31}x_{21} &= x_{21}x_{31}x_{32}. \end{aligned}$$

The next lemma gives a free product decomposition for this group, which will play an important role in the proof that  $vP_3$  is 1-formal.

**Lemma 2.4.** *There is a free product decomposition  $vP_3 \cong \overline{P}_4 * \mathbb{Z}$ .*

*Proof.* As shown in [7], if we set  $a_1 = x_{13}x_{23}$ ,  $b_1 = x_{13}x_{12}$ ,  $b_2 = x_{21}x_{31}$ ,  $a_2 = x_{32}x_{31}$ ,  $c_1 = x_{13}x_{31}$ ,  $c_2 = x_{13}$ , then there is a free product decomposition,  $vP_3 \cong G_3 * \mathbb{Z}$ , where  $G_3$  is generated by  $\{a_1, a_2, b_1, b_2, c_1\}$ , subject to the relations

$$[a_1, b_1] = [a_2, b_2] = 1, \quad b_1^{c_1} = b_1^{a_2}, \quad a_1^{c_1} = a_1^{b_2}, \quad b_2^{c_1} = b_2^{a_1 b_2}, \quad a_2^{c_1} = a_2^{b_1 a_2},$$

where  $y^x = x^{-1}yx$ . Replacing the generators in the presentation of  $G_3$  by  $x_1, x_2, x_3, x_4, x_5^{-1}$ , respectively, and simplifying the relations, we obtain a new presentation for the group  $G_3$ , with generators  $x_1, \dots, x_5$  and relations

$$x_1x_3 = x_3x_1, \quad x_2x_4 = x_4x_2, \quad x_5x_3x_2 = x_3x_2x_5 = x_2x_5x_3, \quad x_1x_4x_5 = x_4x_5x_1 = x_5x_1x_4.$$

On the other hand, as noted for instance in [11], the group  $\overline{P}_4$  has a presentation with generators  $z_1, \dots, z_5$  and relations

$$z_2z_3 = z_3z_2, \quad z_2^{-1}z_4z_2z_1 = z_1z_2^{-1}z_4z_2, \quad z_5z_3z_1 = z_3z_1z_5 = z_1z_5z_3, \quad z_5z_4z_2 = z_4z_2z_5 = z_2z_5z_4.$$

Define a homomorphism  $\phi: G_3 \rightarrow \overline{P}_4$  by sending  $x_1 \mapsto z_2, x_2 \mapsto z_1, x_3 \mapsto z_3, x_4 \mapsto z_2^{-1}z_4z_2$  and  $x_5 \mapsto z_5$ . A routine check shows that  $\phi$  is a well-defined homomorphism, with inverse  $\psi: \overline{P}_4 \rightarrow G_3$  sending  $z_1 \mapsto x_2, z_2 \mapsto x_1, z_3 \mapsto x_3, z_4 \mapsto x_1x_4x_1^{-1}$ , and  $z_5 \mapsto x_5$ . This completes the proof.  $\square$

As a quick application of this lemma, we obtain the following corollary, which was first proved by Bardakov et al. [7] using a different method.

**Corollary 2.5.** *The pure virtual braid group  $vP_3$  is a residually torsion-free nilpotent group.*

*Proof.* It is readily seen that two groups  $G_1$  and  $G_2$  are residually torsion-free nilpotent if and only their direct product,  $G_1 \times G_2$ , is residually torsion-free nilpotent. Now, as shown by Falk and Randell in [19], the pure braid groups  $P_n$  are residually torsion-free nilpotent. Hence, the subgroup  $\overline{P}_4 \subset P_4$  is also residually torsion-free nilpotent.

On the other hand, Malcev [29] showed that if  $G_1$  and  $G_2$  are residually torsion-free nilpotent groups, then the free product  $G_1 * G_2$  is also residually torsion-free nilpotent. The claim follows from the decomposition  $vP_3 \cong \overline{P}_4 * \mathbb{Z}$ .  $\square$

A more general question was asked by Bardakov and Bellingeri in [5]: Are the groups  $vP_n$  or  $vP_n^+$  residually torsion-free nilpotent?

### 3. COHOMOLOGY RINGS AND HILBERT SERIES

In this section we discuss what is known about the cohomology rings of the pure (virtual) braid groups, and the corresponding Hilbert series.

**3.1. Hilbert series and generating functions.** Recall that the (ordinary) *generating function* for a sequence of power series  $\mathbf{P} = \{p_n(t)\}_{n \geq 1}$  is defined by  $F(u, t) := \sum_{n=0}^{\infty} p_n(t)u^n$ . Likewise, the *exponential generating function* for  $\mathbf{P}$  is defined by  $E(u, t) := \sum_{n=0}^{\infty} p_n(t)\frac{u^n}{n!}$ .

Now let  $\mathbf{G} = \{G_n\}_{n \geq 1}$  be a sequence of groups admitting classifying spaces  $K(G_n, 1)$  with finitely many cells in each dimension. We then define the exponential generating function for the corresponding Poincaré polynomials by

$$(7) \quad \text{Poin}(\mathbf{G}, u, t) := 1 + \sum_{n=1}^{\infty} \text{Poin}(G_n, t) \frac{u^n}{n!}.$$

In particular, if we set  $t = -1$ , we obtain the exponential generating function for the Euler characteristics of the groups  $G_n$ , denoted by  $\chi(\mathbf{G})$ .

For instance, the Poincaré polynomial of a free group of rank  $n$  is  $\text{Poin}(F_n, t) = 1 + nt$ . Thus, the exponential generating function for the sequence  $\mathbf{F} = \{F_n\}_{n \geq 1}$  is  $\text{Poin}(\mathbf{F}, u, t) = (1 + tu)e^u$ .

**3.2. Pure braid groups.** A classifying space for the pure braid group  $P_n$  is the configuration space  $\text{Conf}(\mathbb{C}, n)$  of  $n$  distinct points on the complex line. This space has the homotopy type of a finite,  $(n - 1)$ -dimensional CW-complex. As shown by Arnold in [2], the cohomology algebra  $A_n = H^*(P_n; \mathbb{C})$  is the skew-commutative ring generated by degree 1 elements  $a_{ij}$  ( $1 \leq i < j \leq n$ ), subject to the relations

$$(8) \quad a_{ik}a_{jk} = a_{ij}(a_{jk} - a_{ik}) \text{ for } i < j < k.$$

Clearly, this algebra is quadratic. In fact,  $A_n$  is a Koszul algebra, that is to say,  $\text{Ext}_{A_n}^i(\mathbb{C}, \mathbb{C})_j = 0$  for  $i \neq j$ . Furthermore, the Poincaré polynomial of  $\text{Conf}(\mathbb{C}, n)$ , or, equivalently, the Hilbert series of  $A_n$ , is given by

$$(9) \quad \text{Poin}(P_n, t) = \prod_{k=1}^{n-1} (1 + kt) = \sum_{i=0}^{n-1} c(n, n-i) t^i,$$

where  $c(n, m)$  are the (unsigned) Stirling numbers of the first kind, counting the number of permutations of  $n$  elements which contain exactly  $m$  permutation cycles.

**Proposition 3.1.** *The exponential generating function for the Poincaré polynomials of the pure braid groups  $P_n$  is given by*

$$\text{Poin}(\mathbf{P}, u, t) = \exp\left(-\frac{\log(1 - tu)}{t}\right).$$

*Proof.* It is known (see, e.g. [39]) that the exponential generating function for the unsigned Stirling numbers  $c(n, k)$  is given by

$$(10) \quad \exp(-x \cdot \log(1 - z)) = \sum_{n=0}^{\infty} \sum_{k=0}^n c(n, k) x^k \frac{z^n}{n!}.$$

Setting  $x = t^{-1}$  and  $z = tu$ , we obtain

$$(11) \quad \exp\left(-\frac{\log(1 - tu)}{t}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^n c(n, k) t^{-k} \frac{(tu)^n}{n!} = 1 + \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} c(n, n-i) t^i \frac{u^n}{n!},$$

where we used  $c(0, 0) = 1$  and  $c(n, 0) = 0$  for  $n \geq 1$ . This completes the proof.  $\square$

**3.3. Pure virtual braid groups.** In [8], Bartholdi et al. describe classifying spaces for the pure virtual braid group  $vP_n$  and  $vP_n^+$ . Let us note here that both these spaces are finite,  $(n - 1)$ -dimensional CW-complexes.

The following theorem provides presentations for the cohomology algebras of the pure virtual braid groups and their upper triangular subgroups.

**Theorem 3.2** ([8, 27]). *For each  $n \geq 2$ , the following hold.*

- (1) *The cohomology algebra  $A_n = H^*(vP_n; \mathbb{C})$  is the skew-commutative algebra generated by degree 1 elements  $a_{ij}$  ( $1 \leq i \neq j \leq n$ ) subject to the relations  $a_{ij}a_{ik} = a_{ij}a_{jk} - a_{ik}a_{kj}$ ,  $a_{ik}a_{jk} = a_{ij}a_{jk} - a_{ji}a_{ik}$ , and  $a_{ij}a_{ji} = 0$  for  $i, j, k$  all distinct.*
- (2) *The cohomology algebra  $A_n^+ = H^*(vP_n^+; \mathbb{C})$  is the skew-commutative algebra generated by degree 1 elements  $a_{ij}$  ( $1 \leq i \neq j \leq n$ ), subject to the relations  $a_{ij} = -a_{ji}$  and  $a_{ij}a_{jk} = a_{jk}a_{ki}$  for  $i \neq j \neq k$ .*

**Corollary 3.3.** *The cohomology algebra  $H^*(vP_n^+; \mathbb{C})$  has a simplified presentation with generators  $e_{ij}$  in degree 1 for  $1 \leq i < j \leq n$ , and relations  $e_{ij}(e_{ik} - e_{jk})$  and  $(e_{ij} - e_{ik})e_{jk}$  for  $i < j < k$ .*

*Proof.* Let  $\tilde{A}_n^+$  be the algebra given by the above presentation. The morphism  $\phi: A_n^+ \rightarrow \tilde{A}_n^+$  defined by  $\phi(a_{ij}) = e_{ij}$  for  $i < j$  and  $\phi(a_{ij}) = -e_{ij}$  for  $i > j$  is easily checked to be an isomorphism.  $\square$

In [8], Bartholdi et al. also showed that both  $A_n$  and  $A_n^+$  are Koszul algebras, and computed the Hilbert series of these graded algebras, as follows:

$$(12) \quad \text{Poin}(vP_n, t) = \sum_{i=0}^{n-1} L(n, n-i) t^i, \quad \text{Poin}(vP_n^+, t) = \sum_{i=0}^{n-1} S(n, n-i) t^i.$$

Here  $L(n, n-i)$  are the Lah numbers, i.e., the number of ways of partitioning  $[n]$  into  $n-i$  nonempty ordered subsets, while  $S(n, n-i)$  are the Stirling numbers of the second kind, i.e., the number of ways of partitioning  $[n]$  into  $n-i$  nonempty (unordered) sets. Explicitly,

$$(13) \quad L(n, n-i) = \binom{n-1}{i} \frac{n!}{(n-i)!},$$

$$(14) \quad S(n, n-i) = \frac{1}{(n-i)!} \sum_{j=0}^{n-i-1} (-1)^j \binom{n-i}{j} (n-i-j)^n.$$

The polynomial  $\text{Poin}(vP_n^+, t)$  is the rank-generating function for the partition lattice  $\Pi_n$ , see e.g. [39, Exercise 3.10.4]. All the roots of such a polynomial are negative real numbers.

**Proposition 3.4** ([8]). *The exponential generating function for the polynomials  $\text{Poin}(vP_n, t)$  and  $\text{Poin}(vP_n^+, t)$  are given by*

$$\text{Poin}(\mathbf{vP}, u, t) = \exp\left(\frac{u}{1-tu}\right), \quad \text{Poin}(\mathbf{vP}^+, u, t) = \exp\left(\frac{\exp(tu) - 1}{t}\right).$$

Finally, let us note that Dies and Nicas [15] showed that the Euler characteristic of  $vP_n$  is non-zero for all  $n \geq 2$ , while the Euler characteristic of  $vP_n^+$  is non-zero for all  $n \geq 3$ , with one possible exception (and no exception if Wilf's conjecture is true).

#### 4. RESONANCE VARIETIES

In this section we study the resonance varieties of the pure (virtual) braid groups. We start with a discussion of how resonance behaves with respect to products and wedges.

**4.1. Resonance varieties of graded algebras.** Let  $A = \bigoplus_{i \geq 0} A^i$  be a graded, graded-commutative  $\mathbb{C}$ -algebra. We will assume throughout that  $A$  is connected (i.e.,  $A^0 = \mathbb{C}$ ), and locally finite (i.e., the Betti numbers  $b_i := \dim A^i$  are finite, for all  $i \geq 0$ ). By definition, the (degree  $i$ , depth  $d$ ) *resonance varieties* of  $A$  are the algebraic sets

$$(15) \quad \mathcal{R}_d^i(A) = \{a \in A^1 \mid b_i(A, a) \geq d\},$$

where  $(A, a)$  is the cochain complex (known as the *Aomoto complex*) with differentials  $\delta_a^i : A^i \rightarrow A^{i+1}$  given by  $\delta_a^i = a \cdot u$ , and  $b_i(A, a) := \dim H^i(A, a)$ .

Observe that  $b_i(A, 0) = b_i(A)$ . Thus,  $\mathcal{R}_d^i(A)$  is empty if either  $d > b_i$  or  $d \geq 0$  and  $b_i = 0$ . Furthermore,  $0 \in \mathcal{R}_d^i(A)$  if and only if  $d \leq b_i$ . In degree zero, we have that  $\mathcal{R}_d^0(A) = \{0\}$  for  $d = 1$  and  $\mathcal{R}_d^0(A) = \emptyset$  for  $d \geq 2$ . We use the convention that  $\mathcal{R}_d^i(A) = A^1$  for  $d \leq 0$ . The following simple lemma will be useful in computing the resonance varieties of the algebra  $A = H^*(vP_3, \mathbb{C})$ .

**Lemma 4.1.** *Suppose  $A^i \neq 0$  for  $i \leq 2$  and  $A^i = 0$  for  $i \geq 3$ . Then  $\mathcal{R}_d^2(A) = \mathcal{R}_{d-\chi}^1(A)$  for  $d \leq b_2$ , where  $\chi = 1 - b_1 + b_2$  is the Euler characteristic of  $A$ .*

*Proof.* By the above discussion,  $0 \in \mathcal{R}_d^2(A)$  if and only if  $d \leq b_2$ . But this is equivalent to  $0 \in \mathcal{R}_{d-\chi}^1(A)$ , since  $d - \chi \leq b_2 - \chi \leq b_1 - 1$ . Now let  $a \in A^1 \setminus \{0\}$ . Then  $b_2(A, a) = b_1(A, a) + \chi$ . Hence,  $a \in \mathcal{R}_d^2(A)$  if and only if  $a \in \mathcal{R}_{d-\chi}^1(A)$ , and we are done.  $\square$

We will be mostly interested here in the degree 1 resonance varieties,  $\mathcal{R}_d^1(A)$ . Equations for these varieties can be obtained as follows (see for instance [41]). Let  $\{e_1, \dots, e_n\}$  be a basis for the complex vector space  $A^1 = H^1(G; \mathbb{C})$ , and let  $\{x_1, \dots, x_n\}$  be the dual basis for  $A_1 = H_1(G; \mathbb{C})$ . Identifying the symmetric algebra  $\text{Sym}(A_1)$  with the polynomial ring  $S = \mathbb{C}[x_1, \dots, x_n]$ , we obtain a cochain complex of free  $S$ -modules,

$$(16) \quad A^0 \otimes_{\mathbb{C}} S \xrightarrow{\delta^0} A^1 \otimes_{\mathbb{C}} S \xrightarrow{\delta^1} A^2 \otimes_{\mathbb{C}} S \xrightarrow{\delta^2} \dots,$$

with differentials given by  $\delta^i(u \otimes 1) = \sum_{j=1}^n e_j u \otimes x_j$  for  $u \in A^i$  and extended by  $S$ -linearity. The resonance variety  $\mathcal{R}_d^1(A)$ , then, is the zero locus of the ideal of codimension  $d$  minors of the matrix  $\delta^1$ .

Now suppose  $X$  is a connected, finite-type CW-complex. One defines then the resonance varieties of  $X$  to be the sets  $\mathcal{R}_d^i(X) := \mathcal{R}_d^i(H^*(X, \mathbb{C}))$ . Likewise, the resonance varieties of a group  $G$  admitting a finite-type classifying space are defined as  $\mathcal{R}_d^i(G) := \mathcal{R}_d^i(H^*(G, \mathbb{C}))$ .

**4.2. Resonance varieties of products and coproducts.** The next two results are generalizations of Propositions 13.1 and 13.3 from [35]. We will use these results to compute the resonance varieties of the group  $vP_3$ .

**Proposition 4.2.** *Let  $A = B \otimes C$  be the product of two connected, finite-type graded algebras. Then, for all  $i \geq 1$ ,*

$$\begin{aligned}\mathcal{R}_d^1(B \otimes C) &= \mathcal{R}_d^1(B) \times \{0\} \cup \{0\} \times \mathcal{R}_d^1(C), \\ \mathcal{R}_1^i(B \otimes C) &= \bigcup_{s+t=i} \mathcal{R}_1^s(B) \times \mathcal{R}_1^t(C).\end{aligned}$$

*Proof.* Let  $a = (a_1, a_2)$  be an element in  $A^1 = B^1 \oplus C^1$ . The cochain complex  $(A, a)$  splits as a tensor product of cochain complexes,  $(B, a_1) \otimes (C, a_2)$ . Therefore,

$$(17) \quad b_i(A, a) = \sum_{s+t=i} b_s(B, a_1) b_t(C, a_2),$$

and the second formula follows. In particular, we have  $b_1(A, (0, 0)) = b_1(B, 0) + b_1(C, 0)$ ,  $b_1(A, (0, a_2)) = b_1(C, a_2)$  if  $a_2 \neq 0$ ,  $b_1(A, (a_1, 0)) = b_1(B, a_1)$  if  $a_1 \neq 0$ , and  $b_1(A, a) = 0$  if  $a_1 \neq 0$  and  $a_2 \neq 0$ . The first formula now easily follows.  $\square$

**Proposition 4.3.** *Let  $A = B \vee C$  be the coproduct of two connected, finite-type graded algebras. Then, for all  $i \geq 1$ ,*

$$\begin{aligned}\mathcal{R}_d^1(B \vee C) &= \left( \bigcup_{j+k=d-1} (\mathcal{R}_j^1(B) \setminus \{0\}) \times (\mathcal{R}_k^1(C) \setminus \{0\}) \right) \cup \\ &\quad (\{0\} \times \mathcal{R}_s^1(C)) \cup (\mathcal{R}_t^1(B) \times \{0\}), \\ \mathcal{R}_d^i(B \vee C) &= \bigcup_{j+k=d} \mathcal{R}_j^i(B) \times \mathcal{R}_k^i(C), \quad \text{if } i \geq 2,\end{aligned}$$

where  $s = d - \dim B^1$  and  $t = d - \dim C^1$ .

*Proof.* Pick an element  $a = (a_1, a_2)$  in  $A^1 = B^1 \oplus C^1$ . The Aomoto complex of  $A$  splits (in positive degrees) as a direct sum of chain complexes,  $(A^+, a) \cong (B^+, a_1) \oplus (C^+, a_2)$ . We then have formulas relating the Betti numbers of the respective Aomoto complexes:

$$b_i(A, a) = \begin{cases} b_i(B, a_1) + b_i(C, a_2) + 1 & \text{if } i = 1, \text{ and } a_1 \neq 0, a_2 \neq 0, \\ b_i(B, a_1) + b_i(C, a_2) & \text{otherwise.} \end{cases}$$

The claim follows by a case-by-case analysis of the above formula.  $\square$

**4.3. Pure braid groups.** Since  $P_n$  admits a classifying space of dimension  $n - 1$ , the resonance varieties  $\mathcal{R}_d^i(P_n)$  are empty for  $i \geq n$ . In degree  $i = 1$ , the resonance varieties  $\mathcal{R}_d^i(P_n)$  are either trivial, or a union of 2-dimensional subspaces.

**Proposition 4.4** ([13]). *The first resonance variety of the pure braid group  $P_n$  has decomposition into irreducible components given by*

$$\mathcal{R}_1^1(P_n) = \bigcup_{1 \leq i < j < k \leq n} L_{ijk} \cup \bigcup_{1 \leq i < j < k < l \leq n} L_{ijkl},$$

where

$$\begin{aligned} L_{ijk} &= \{x_{ij} + x_{ik} + x_{jk} = 0 \text{ and } x_{st} = 0 \text{ if } \{s, t\} \not\subset \{i, j, k\}\}, \\ L_{ijkl} &= \left\{ \begin{array}{l} \sum_{\{p,q\} \subset \{i,j,k,l\}} x_{pq} = 0, \quad x_{ij} = x_{kl}, \quad x_{jk} = x_{il}, \quad x_{ik} = x_{jl}, \\ x_{st} = 0 \text{ if } \{s, t\} \not\subset \{i, j, k, l\} \end{array} \right\}. \end{aligned}$$

Furthermore,  $\mathcal{R}_d^1(P_n) = \{0\}$  for  $2 \leq d \leq \binom{n}{2}$ , and  $\mathcal{R}_d^1(P_n) = \emptyset$  for  $d > \binom{n}{2}$ .

Recall that  $P_n \cong \overline{P}_n \times \mathbb{Z}$ , where  $\overline{P}_n$  is the quotient of  $P_n$  by its (infinite cyclic) center. Thus, the resonance varieties of the group  $\overline{P}_n$  can be described in a similar manner, using Proposition 4.2.

**4.4. Resonance varieties of  $vP_3$ .** A partial computation of the resonance varieties  $\mathcal{R}_d^i(vP_3)$  was done in a preprint version of [7] for  $i = 1$  and  $d = 1, 5, 6$ , as well as  $i = 2$  and  $d = 2, 6$ . We use the preceding discussion to give a complete computation of all these varieties.

**Proposition 4.5.** *For  $d \geq 1$ , the resonance varieties of the pure virtual braid group  $vP_3$  are given by*

$$\mathcal{R}_d^i(vP_3) \cong \begin{cases} \mathcal{R}_{d-1}^1(\overline{P}_4) \times \mathbb{C}, & \text{for } i = 1, d \leq 5 \\ \{0\} & \text{for } i = 1, d = 6, \\ \mathcal{R}_{d-2}^1(\overline{P}_4) \times \mathbb{C} & \text{for } i = 2, d \leq 6 \\ \emptyset & \text{otherwise.} \end{cases}$$

Consequently,  $\mathcal{R}_1^1(vP_3) = \mathbb{C}^6$ , while  $\mathcal{R}_2^1(vP_3)$  is a union of five 3-dimensional subspaces, pairwise intersecting in the 1-dimensional subspace  $\mathcal{R}_3^1(vP_3) = \mathcal{R}_4^1(vP_3) = \mathcal{R}_5^1(vP_3)$ .

*Proof.* By Lemma 2.4, we have an isomorphism  $vP_3 \cong \overline{P}_4 * \mathbb{Z}$ , which yields an isomorphism  $H^1(vP_3; \mathbb{C}) \cong H^1(\overline{P}_4; \mathbb{C}) \oplus \mathbb{C}$ . Under this identification, Proposition 4.3 shows that  $\mathcal{R}_d^1(vP_3) \cong \mathcal{R}_{d-1}^1(\overline{P}_4) \times \mathbb{C}$  for  $d \leq 5$ , and  $\mathcal{R}_6^1(vP_3) = \{0\}$ .

The same proposition also shows that  $\mathcal{R}_d^2(vP_3) \cong \mathcal{R}_d^2(\overline{P}_4) \times \mathbb{C}$ . On the other hand, by Lemma 4.1, we have that  $\mathcal{R}_d^2(\overline{P}_4) = \mathcal{R}_{d-2}^1(\overline{P}_4)$  for  $d \leq 6$ , since  $\overline{P}_4$  admits a 2-dimensional classifying space, and  $\chi(\overline{P}_4) = 2$ . Finally, the description of the resonance varieties  $\mathcal{R}_d^1(vP_3)$  for  $d \leq 6$  follows from Proposition 4.4.  $\square$

Let  $a_{12}, a_{13}, a_{23}, a_{21}, a_{31}, a_{32}$  be the basis of  $H^1(vP_3, \mathbb{C})$  specified in Theorem 3.2, and let  $x_{ij}$  the corresponding coordinate functions on this affine space. Tracing through the isomorphisms  $H^1(vP_3, \mathbb{C}) \cong H^1(\overline{P}_4, \mathbb{C}) \times \mathbb{C} \cong H^1(P_4, \mathbb{C})$ , we see that the components of  $\mathcal{R}_2^1(vP_3)$  have equations

$$\begin{aligned} \{x_{12} - x_{23} = x_{12} + x_{32} = x_{12} + x_{21} = 0\}, \quad &\{x_{13} + x_{23} = x_{12} + x_{32} = x_{21} + x_{31} = 0\}, \\ \{x_{13} + x_{23} = x_{13} - x_{32} = x_{13} + x_{31} = 0\}, \quad &\{x_{12} + x_{13} = x_{12} + x_{21} = x_{12} - x_{31} = 0\}, \\ \{x_{12} + x_{13} = x_{23} + x_{21} = x_{31} + x_{32} = 0\}, \quad & \end{aligned}$$

while their common intersection is the line  $\{x_{12} = -x_{21} = -x_{13} = x_{31} = x_{23} = -x_{32}\}$ .

**4.5. Resonance varieties of  $vP_4^+$ .** We now switch to the upper-triangular group  $vP_4^+$ , and compute its degree 1 resonance varieties. Let  $e_{12}, e_{13}, e_{23}, e_{14}, e_{24}, e_{34}$  be the basis of  $H^1(vP_4^+, \mathbb{C})$  specified in Corollary 3.3, and let  $x_{ij}$  be the corresponding coordinate functions on this affine space.

**Lemma 4.6.** *The depth 1 resonance variety  $\mathcal{R}_1^1(vP_4^+)$  is the irreducible, 4-dimensional subvariety of degree 6 inside  $H^1(vP_4^+, \mathbb{C}) = \mathbb{C}^6$  defined by the equations*

$$\begin{cases} x_{12}x_{24}(x_{13} + x_{23}) + x_{13}x_{34}(x_{12} - x_{23}) - x_{24}x_{34}(x_{12} + x_{13}) = 0, \\ x_{12}x_{23}(x_{14} + x_{24}) + x_{12}x_{34}(x_{23} - x_{14}) + x_{14}x_{34}(x_{23} + x_{24}) = 0, \\ x_{13}x_{23}(x_{14} + x_{24}) + x_{14}x_{24}(x_{13} + x_{23}) + x_{34}(x_{13}x_{23} - x_{14}x_{24}) = 0, \\ x_{12}(x_{13}x_{14} - x_{23}x_{24}) + x_{34}(x_{13}x_{23} - x_{14}x_{24}) = 0. \end{cases}$$

The depth 2 resonance variety  $\mathcal{R}_2^1(vP_4^+)$  consists of 13 lines in  $\mathbb{C}^6$ , spanned by the vectors

$$\begin{aligned} &e_{12}, \quad e_{13}, \quad e_{23}, \quad e_{14}, \quad e_{24}, \quad e_{34}, \quad e_{12} - e_{13} + e_{23}, \\ &e_{12} - e_{14} + e_{24}, \quad e_{13} - e_{14} + e_{34}, \quad e_{23} - e_{24} + e_{34}, \\ &e_{13} - e_{23} - e_{14} + e_{24}, \quad e_{12} + e_{23} - e_{14} + e_{34}, \quad e_{12} - e_{13} + e_{24} - e_{34}. \end{aligned}$$

Finally,  $\mathcal{R}_d^1(vP_4^+) = \{0\}$  if  $3 \leq d \leq 6$ , and  $\mathcal{R}_d^1(vP_4^+) = \emptyset$  if  $d \geq 7$ .

*Proof.* Let  $A = H^*(vP_4^+, \mathbb{C})$  be the cohomology algebra of  $vP_4^+$ , as described in Corollary 3.3. The differential  $\delta^1: A^1 \otimes S \rightarrow A^2 \otimes S$  in the cochain complex (16) is then given by

$$\delta^1 = \begin{pmatrix} -x_{34} & 0 & 0 & 0 & 0 & x_{12} \\ -x_{13} - x_{23} & x_{12} - x_{23} & x_{12} + x_{13} & 0 & 0 & 0 \\ 0 & -x_{24} & 0 & 0 & x_{13} & 0 \\ 0 & 0 & -x_{14} & x_{23} & 0 & 0 \\ -x_{14} - x_{24} & 0 & 0 & x_{12} - x_{24} & x_{12} + x_{14} & 0 \\ 0 & -x_{14} - x_{34} & 0 & x_{13} - x_{34} & 0 & x_{13} + x_{14} \\ 0 & 0 & -x_{24} - x_{34} & 0 & x_{23} - x_{34} & x_{23} + x_{24} \end{pmatrix}.$$

Computing with the aid of Macaulay2 [28] the elementary ideals of this matrix and finding their primary decomposition leads to the stated conclusions.  $\square$

The degree 1, depth 1 resonance varieties of the virtual pure braid groups  $vP_n^+$  and  $vP_n$  can be computed in a similar fashion, at least for small values of  $n$ . For instance,  $\mathcal{R}_1^1(vP_5^+)$  has 15 irreducible components of dimension 4, while  $\dim \mathcal{R}_1^1(vP_n^+) = n - 1$  for  $n = 6, 7, 8$ . In general, though, these varieties are not equidimensional. For example,  $\mathcal{R}_1^1(vP_4)$  has seven irreducible components of dimension 4, three irreducible components of dimension 5, and four irreducible components of dimension 6.

## 5. GRADED LIE ALGEBRAS AND GRADED FORMALITY

We now discuss the holonomy and associated graded Lie algebras of the pure (virtual) braid groups. For relevant background, we refer to [42] and references therein.

**5.1. Associated graded Lie algebras.** Let  $G$  be a finitely generated group. There are several Lie algebras associated to such a group. The most classical one is the associated graded Lie algebra (over  $\mathbb{C}$ ),

$$(18) \quad \text{gr}(G) := \bigoplus_{k \geq 1} (\Gamma_k G / \Gamma_{k+1} G) \otimes_{\mathbb{Z}} \mathbb{C},$$

where  $\{\Gamma_k G\}_{k \geq 1}$  is the *lower central series*, defined inductively by  $\Gamma_1 G = G$  and  $\Gamma_{k+1} G = [\Gamma_k G, G]$ ,  $k \geq 1$ , and the Lie bracket  $[x, y]$  is induced from the group commutator  $[x, y] = xyx^{-1}y^{-1}$ . We denote by  $U = U(\text{gr}(G))$  the universal enveloping algebra of  $\text{gr}(G)$ . By the Poincaré–Birkhoff–Witt theorem, we have that

$$(19) \quad \prod_{k=1}^{\infty} (1 - t^k)^{-\phi_k(G)} = \text{Hilb}(U, t).$$

The next two lemmas give explicit ways to compute the LCS ranks of a group  $G$ , under a common rationality hypothesis for the Hilbert series of the graded Lie algebra  $U$ .

**Lemma 5.1.** *Suppose there is a polynomial  $f(t) = 1 + \sum_{i=1}^n b_i t^i \in \mathbb{Z}[t]$  such that*

$$(20) \quad \text{Hilb}(U(\text{gr}(G)), -t) \cdot f(t) = 1.$$

*Then the LCS ranks of  $G$  are given by*

$$(21) \quad \phi_k(G) = \frac{1}{k} \sum_{d|k} \mu\left(\frac{k}{d}\right) \left[ \sum_{m_1+2m_2+\dots+nm_n=d} (-1)^{s_n} d(m!) \prod_{j=1}^n \frac{(b_j)^{m_j}}{(m_j)!} \right],$$

where  $0 \leq m_j \in \mathbb{Z}$ ,  $s_n = \sum_{i=1}^{[n/2]} m_{2i}$ ,  $m = \sum_{i=1}^n m_i - 1$  and  $\mu$  is the Möbius function.

*Proof.* From formula (19) and assumption (20), we have that

$$(22) \quad \prod_{k=1}^{\infty} (1 - t^k)^{\phi_k(G)} = 1 + \sum_{i=1}^n b_i (-t)^i.$$

Taking logarithms on both sides, we find that

$$(23) \quad \sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \phi_s(G) \frac{t^{sj}}{j} = \sum_{w=1}^{\infty} \frac{1}{w} \left( - \sum_{i=1}^n b_i (-t)^i \right)^w.$$

Comparing the coefficients of  $t^k$  on each side gives

$$(24) \quad \sum_{d|k} \phi_d(G) \frac{d}{k} = \sum_{m_1+2m_2+\dots+nm_n=k} (-1)^{s_n} (m!) \prod_{j=1}^n \frac{(b_j)^{m_j}}{(m_j)!},$$

where  $s_n = \sum_{i=1}^{[n/2]} m_{2i}$  and  $m = \sum_{i=1}^n m_i - 1$ . Finally, multiplying both sides by  $k$  and using the Möbius inversion formula yields the desired formula.  $\square$

An alternative way of computing the LCS ranks of a group  $G$  satisfying the assumptions from Lemma 5.1 was given by Weigel in [46].

**Lemma 5.2** ([46]). *Suppose there is a polynomial  $f(t) = 1 + \sum_{i=1}^n b_i t^i \in \mathbb{Z}[t]$  such that  $\text{Hilb}(U(\text{gr}(G), -t)) \cdot f(t) = 1$ . Let  $z_1, \dots, z_n$  be the (complex) roots of  $f(-t)$ . Then the LCS ranks of  $G$  are given by*

$$(25) \quad \phi_k(G) = \frac{1}{k} \sum_{1 \leq i \leq n} \sum_{d|k} \mu\left(\frac{k}{d}\right) \frac{1}{z_i^d}.$$

**5.2. Holonomy Lie algebras.** Let  $A = \bigoplus_{i \geq 0} A_i$  be a graded, graded-commutative, algebra over  $\mathbb{C}$ . We shall assume that  $A_0 = \mathbb{C}$ , and  $\dim A_1 < \infty$ . Denote by  $A_i^*$  the dual vector space to  $A_i$ . We let  $\partial_A: A_2^* \rightarrow A_1^* \wedge A_1^*$  be the dual of the product  $A_1 \wedge A_1 \rightarrow A_2$ . Let  $\mathbf{L}(A_1^*)$  be the free, complex Lie algebra on  $A_1^*$ , with the degree 2 piece identified with  $A_1^* \wedge A_1^*$ . We define the *holonomy Lie algebra* of  $A$  as the quotient

$$(26) \quad \mathfrak{h}(A) = \mathbf{L}(A_1^*) / \langle \text{im } \partial_A \rangle.$$

Now let  $G$  be a finitely generated group. The holonomy Lie algebra of  $G$ , denoted by  $\mathfrak{h}(G)$ , is defined to be the holonomy of its cohomology algebra  $A = H^*(G; \mathbb{C})$ . The identification of the  $\mathbb{C}$ -vector space  $H_{\mathbb{C}} := H_1(G, \mathbb{C})$  with  $\text{gr}_1(G) = G/G' \otimes \mathbb{C}$  extends to a surjective morphism of graded Lie algebras,  $\mathbf{L}(H_{\mathbb{C}}) \twoheadrightarrow \text{gr}(G)$ , which sends  $\text{im}(\partial_G) \subset \mathbf{L}^2(H_{\mathbb{C}})$  to zero. In this fashion, we obtain an epimorphism

$$(27) \quad \Psi: \mathfrak{h}(G) \longrightarrow \text{gr}(G),$$

which is readily seen to be an isomorphism in degrees 1 and 2. Thus, the holonomy Lie algebra  $\mathfrak{h}(G)$  can be viewed as the quadratic closure of  $\text{gr}(G)$ .

The group  $G$  is said to be *graded-formal* if  $\Psi$  is, in fact, an isomorphism of graded Lie algebras. As noted in [42], the group  $G$  is graded-formal if and only if  $\text{gr}(G)$  is quadratic. Write

$$(28) \quad \phi_k(G) = \dim \text{gr}_k(G) \quad \text{and} \quad \bar{\phi}_k(G) = \dim \mathfrak{h}_k(G).$$

Clearly  $\phi_k(G) \leq \bar{\phi}_k(G)$ , with equality for  $k = 1$  and 2. Moreover, the group  $G$  is graded-formal if and only if  $\phi_k(G) = \bar{\phi}_k(G)$  for all  $k \geq 1$ .

**Proposition 5.3.** *Suppose the group  $G$  is graded-formal, and its cohomology algebra,  $A = H^*(G; \mathbb{C})$ , is Koszul. Then  $\text{Hilb}(U(\text{gr}(G)), -t) \cdot \text{Hilb}(A, t) = 1$ .*

*Proof.* Let  $U = U(\text{gr}(G))$ , and let  $U^!$  be its quadratic dual. By assumption,  $\text{gr}(G) = \mathfrak{h}(A)$  is a quadratic Lie algebra. Thus,  $U$  is a quadratic algebra. Furthermore, since  $A$  is also quadratic,  $U = U(\mathfrak{h}(A))$  is isomorphic to  $A^!$ , the quadratic dual of  $A$ , see [36].

On the other hand, since  $A$  is Koszul, the Koszul duality formula gives  $\text{Hilb}(A^!, -t) \cdot \text{Hilb}(A, t) = 1$ . The conclusion follows.  $\square$

**Corollary 5.4.** *Suppose the group  $G$  is graded-formal, and its cohomology algebra is Koszul and finite-dimensional. Then the LCS ranks  $\phi_k(G)$  are given by formula (21), where  $b_i = b_i(G)$ .*

**5.3. Mild presentations.** Let  $G$  be a group admitting a finite presentation  $F/R = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ . The *weight* of a word  $r \in F$  is defined as  $\omega(r) = \sup\{k \mid r \in \Gamma_k F\}$ . The image of  $r$  in  $\text{gr}_{\omega(r)}(F)$  is called the *initial form* of  $r$ , and is denoted by  $\text{in}(r)$ .

As before, let  $\mathbf{L}(H_{\mathbb{C}})$  be the free Lie algebra on the  $\mathbb{C}$ -vector space  $H_{\mathbb{C}} = H_1(G, \mathbb{C})$ , and write

$$(29) \quad \mathfrak{L}(G) := \mathbf{L}(H_{\mathbb{C}})/J,$$

where  $J$  is the ideal of  $\mathbf{L}(H_{\mathbb{C}})$  generated by  $\{\text{in}(r_1), \dots, \text{in}(r_m)\}$ . Work of Labute [26] shows that  $J/[J, J]$  can be viewed as a  $U(\mathfrak{L}(G))$ -module via the adjoint representation of the Lie algebra  $\mathfrak{L}(G)$ . If the module  $J/[J, J]$  is a free  $U(\mathfrak{L}(G))$ -module on the images of  $\text{in}(r_1), \dots, \text{in}(r_m)$ , the given presentation of  $G$  are called *mild*. If a presentation  $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$  is mild, then the Lie algebra  $\text{gr}(G)$  is isomorphic to  $\mathfrak{L}(G)$ . As shown by Anick in [1], the presentation for  $G$  is mild if and only if

$$(30) \quad \text{Hilb}(U(\mathfrak{L}(G)), t) = \left(1 - nt + \sum_{i=1}^m t^{\omega(r_i)}\right)^{-1}.$$

**Lemma 5.5.** *Let  $G$  be a group admitting a mild presentation  $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$  such that  $r_i \in [F, F]$  for  $1 \leq i \leq m$ . If  $G$  is graded-formal, then the LCS ranks of  $G$  are given by*

$$(31) \quad \phi_k(G) = \frac{1}{k} \sum_{d|k} \mu(k/d) \frac{\left(n + \sqrt{n^2 - 4m}\right)^d + \left(n - \sqrt{n^2 - 4m}\right)^d}{(2m)^d}.$$

Moreover, if the enveloping algebra  $U = U(\text{gr}(G; \mathbb{C}))$  is Koszul, then

$$\text{Hilb}(\text{Ext}_U(\mathbb{C}; \mathbb{C}), t) = 1 + nt + mt^2.$$

*Proof.* Since  $G$  has a mild presentation,  $\text{gr}(G)$  is isomorphic to the Lie algebra  $\mathfrak{L}(G)$  associated to this presentation. Furthermore, since  $G$  is a graded-formal, and all the relators  $r_i$  are commutators, we have that  $\omega(r_i) = 2$  for  $1 \leq i \leq m$ . Using now the Poincaré–Birkhoff–Witt theorem and formula (30), we find that  $\text{Hilb}(U(\text{gr}(G)), t) \cdot (1 - nt + mt^2) = 1$ . Hence, the LCS ranks formula follows from Lemma 5.2.

Now suppose  $U = U(\text{gr}(G))$  is a Koszul algebra. Then  $\text{Ext}_U(\mathbb{C}; \mathbb{C}) = U^!$ , and the expression for the Hilbert series of  $\text{Ext}_U(\mathbb{C}; \mathbb{C})$  follows from (19).  $\square$

**5.4. Graded algebras associated to  $P_n$ ,  $vP_n$  and  $vP_n^+$ .** We start with the pure braid groups  $P_n$ . As shown by Kohno [24] and Falk–Randell [18], the graded Lie algebra  $\text{gr}(P_n)$  is generated by degree 1 elements  $s_{ij}$  for  $1 \leq i \neq j \leq n$ , subjects to the relations

$$(32) \quad s_{ij} = s_{ji}, [s_{jk}, s_{ik} + s_{ij}] = 0, [s_{ij}, s_{kl}] = 0 \text{ for } i \neq j \neq l.$$

In particular, the pure braid group  $P_n$  is graded-formal. The universal enveloping algebra  $U(\text{gr}(P_n))$  is Koszul with Hilbert series  $\prod_{k=1}^{n-1} (1 - kt)^{-1}$ . Using Koszul duality and formulas (9) and (21), we see that formula (1) from the Introduction holds for the groups  $G_n =$

$P_n$ . Alternatively, the ranks  $\phi_k(P_n)$  can be computed from formula (19), as follows (see [24, 18, 36]),

$$(33) \quad \phi_k(P_n) = \sum_{s=1}^{n-1} \phi_k(F_s) = \frac{1}{k} \sum_{s=1}^{n-1} \sum_{d|k} \mu(k/d) s^d.$$

Next, we give a presentation for the holonomy Lie algebras of the pure virtual braid groups, using a method described in [33, 42]. The Lie algebra  $\mathfrak{h}(vP_n)$  is generated by  $r_{ij}$ ,  $1 \leq i \neq j \leq n$ , with relations

$$(34) \quad [r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] + [r_{ik}, r_{jk}] = 0 \text{ for distinct } i, j, k,$$

and  $[r_{ij}, r_{kl}] = 0$  for distinct  $i, j, k, l$ . The Lie algebra  $\mathfrak{h}(vP_n^+)$  is the quotient Lie algebra of  $\mathfrak{h}(vP_n)$  by the ideal generated by  $r_{ij} + r_{ji}$  for distinct  $i \neq j$ . Similarly as Corollary 3.3, the Lie algebra  $\mathfrak{h}(vP_n^+)$  has a simplified presentation with generators  $r_{ij}$  for  $i < j$  and the corresponding relations of  $\mathfrak{h}(vP_n)$ .

**Theorem 5.6** ([8, 27]). *The Lie algebra  $\mathfrak{h}(vP_n)$  is isomorphic to the Lie algebra  $\text{gr}(vP_n)$ . Likewise, the Lie algebra  $\mathfrak{h}(vP_n^+)$  is isomorphic to the Lie algebra  $\text{gr}(vP_n^+)$ .*

The next corollary shows that formula (1) from the Introduction also holds for the virtual pure braid groups.

**Corollary 5.7.** *The LCS ranks of the groups  $G_n = vP_n$  and  $vP_n^+$  are given by*

$$\phi_k(G_n) = \frac{1}{k} \sum_{d|k} \mu\left(\frac{k}{d}\right) \left[ \sum_{m_1+2m_2+\dots+nm_n=d} (-1)^{s_n} d(m!) \prod_{j=1}^n \frac{(b_{n,n-j})^{m_j}}{(m_j)!} \right],$$

where  $m_j$  are non-negative integers,  $s_n = \sum_{i=1}^{[n/2]} m_{2i}$ ,  $m = \sum_{i=1}^n m_i - 1$ , and  $b_{n,j}$  are the Lah numbers for  $G_n = vP_n$  and the Stirling numbers of the second kind for  $G_n = vP_n^+$ .

*Proof.* As mentioned in §3.3, work of Bartholdi et al. [8] and Lee [27] shows that the cohomology algebra  $A = H^*(G_n; \mathbb{C})$  is Koszul. Furthermore, it follows from Theorem 5.6 that  $A \cong U(\text{gr}(G_n))!$ . Hence, by the Koszul duality formula mentioned in Proposition 5.3, we have that  $\text{Hilb}(U(\text{gr}(G_n), -t)) \cdot \text{Poin}(G_n, t) = 1$ , thereby showing that  $G_n$  satisfies the hypothesis of Lemma 5.1. Using now formulas (12) and (21) yields the desired expression for the LCS ranks of the groups  $G_n$ .  $\square$

**Remark 5.8.** The referee asked whether it is possible to compute explicitly the coefficients of the series  $\text{Hilb}(U(\text{gr}(G_n), t)) = 1 / \text{Poin}(G_n, -t)$ , where recall the polynomials  $\text{Poin}(G_n, t)$  are given in (9) and (12). As pointed out by the referee, the classical identity  $\prod_{k=1}^n (1 - kt)^{-1} = \sum_{i \geq 0} S(n+i, n) t^i$ , where  $S(n+i, n)$  are the Stirling numbers of the second kind (see (14) and [39, (1.94c)]), implies that

$$(35) \quad \text{Hilb}(U(\text{gr}(P_{n+1})), t) = \sum_{i \geq 0} S(n+i, n) t^i.$$

Direct computation shows that  $\text{Hilb}(U(\text{gr}(vP_3^+)), t) = \sum_{i \geq 0} a_{2i+2} t^i$ , where  $a_n$  is the  $n$ -th Fibonacci number, while  $\text{Hilb}(U(\text{gr}(vP_3)), t) = \sum_{i \geq 0} \sqrt{6}^i U_i(\sqrt{6}/2) t^i$ , where  $U_n(x)$  is the

$n$ -th Chebyshev polynomial of the second kind. We do not know the corresponding Hilbert series expansions for the groups  $vP_n^+$  and  $vP_n$  with  $n \geq 4$ .

**5.5. Non-mild presentations.** Again, let  $G_n$  denote any one of the pure braid-like groups  $P_n$ ,  $vP_n$ , or  $vP_n^+$ . Recall that  $G_n$  is graded-formal, and  $\text{gr}(G_n) \cong \mathfrak{L}(G_n)$ . However, as we show next, the groups  $G_n$  are not mildly presented, except for small  $n$ .

**Proposition 5.9.** *The pure braid groups  $P_n$  and the pure virtual braid groups  $vP_n$  and  $vP_n^+$  admit mild presentations if and only if  $n \leq 3$ .*

*Proof.* Let  $G_n$  denote any of the aforementioned groups. Then  $G_n$  is a commutator-relators group, and the universal enveloping algebra of the associated graded Lie algebra is Koszul. From formulas (9) and (12), for  $n \leq 3$ , Anick's criterion (30) is satisfied. Hence,  $G_n$  has a mild presentation for  $n \leq 3$ .

Now suppose  $n \geq 4$ . Using formulas (9) and (12) once again, we see that the third Betti numbers of these groups are given by

$$(36) \quad b_3(P_n) = s(n, n-3), \quad b_3(vP_n) = L(n, n-3), \quad b_3(vP_n^+) = S(n, n-3).$$

Thus,  $\dim H^3(G_n, \mathbb{C}) > 0$  for  $n \geq 4$ . The claim now follows from Proposition 5.5 and the fact that  $H^*(G_n, \mathbb{C}) = U(\text{gr}(G_n, \mathbb{C}))^! = \text{Ext}_U(\mathbb{C}, \mathbb{C})$ .  $\square$

## 6. MALCEV LIE ALGEBRAS AND FILTERED FORMALITY

In this section we study the formality properties of the pure virtual braid groups, and prove Theorem 1.2 from the Introduction. We start with a review of the relevant formality notions, following Quillen [38] and Sullivan [45]. For more details and references, we refer to [42].

**6.1. Malcev Lie algebras and 1-formality.** Let  $G$  be a finitely generated group. The group-algebra  $A = \mathbb{C}[G]$  is a Hopf algebra, with comultiplication  $\Delta: A \otimes A \rightarrow A$  given by  $\Delta(g) = g \otimes g$  for  $g \in G$ , and counit the map  $\varepsilon: A \rightarrow \mathbb{C}$  given by  $\varepsilon(g) = 1$ . Let  $\widehat{A} = \varprojlim_r A/I^r$  be the completion of  $A$  with respect to the  $I$ -adic filtration, where  $I = \ker(\varepsilon)$  is the augmentation ideal. The set  $\mathfrak{m}(G)$  of all primitive elements (that is, the set of all  $x \in \widehat{A}$  such that  $\widehat{\Delta}x = x\widehat{\otimes}1 + 1\widehat{\otimes}x$ ), with Lie bracket  $[x, y] = xy - yx$ , and endowed with the induced filtration, is known as the *Malcev Lie algebra* of  $G$ .

Let  $\widehat{\text{gr}}(G)$  be the completion of the associated graded Lie algebra of  $G$  with respect to the lower central series filtration. The group  $G$  is said to be *filtered-formal* if  $\mathfrak{m}(G)$  is isomorphic (as a filtered Lie algebra) to  $\widehat{\text{gr}}(G)$ , or, equivalently, if there exists a morphism of filtered Lie algebras,  $\phi: \mathfrak{m}(G) \rightarrow \widehat{\text{gr}}(G)$ , such that  $\text{gr}_1(\phi)$  is an isomorphism. The group  $G$  is said to be *1-formal* if  $\mathfrak{m}(G) \cong \widehat{\mathfrak{h}}(G)$ . Clearly,  $G$  is 1-formal if and only if it is both graded-formal and filtered-formal.

It is easily seen that  $G$  is 1-formal if  $b_1(G)$  is 0 or 1. Furthermore, if  $F$  is a finitely generated free group, then  $\mathfrak{m}(F)$  is the completed free Lie algebra of the same rank, and thus  $F$  is 1-formal. Other well-known examples of 1-formal groups include the fundamental groups of compact Kähler manifolds ([16]) and the fundamental groups of complements of complex algebraic hypersurfaces ([25]). As noted by Bezrukavnikov [9], the group  $P_{g,n}$

of pure braids on a surface of genus  $g$  is always filtered-formal; it is also graded-formal for  $g \geq 2$  or  $g = 1$  and  $n \leq 2$ , but not for  $g = 1$  and  $n \geq 3$  (see also [17]).

**6.2. Formality, group operations, and resonance.** As shown in [17], the 1-formality property of groups is preserved under (finite) products and the coproducts. We sharpen these results in [42], as follows.

**Theorem 6.1** ([42]). *Let  $G_1$  and  $G_2$  be two finitely generated groups. The following conditions are equivalent.*

- (1)  $G_1$  and  $G_2$  are graded-formal (respectively, filtered-formal, or 1-formal).
- (2)  $G_1 * G_2$  is graded-formal (respectively, filtered-formal, or 1-formal).
- (3)  $G_1 \times G_2$  is graded-formal (respectively, filtered-formal, or 1-formal).

The next theorem shows how the formality properties are preserved under split inclusions.

**Theorem 6.2** ([42]). *Let  $N$  be a subgroup of a finitely generated group  $G$ . Suppose there is a split monomorphism  $\iota: N \rightarrow G$ . If  $G$  is 1-formal, then  $N$  is also 1-formal. A similar statement holds for graded-formality and filtered-formality.*

An important obstruction to 1-formality is provided by the higher-order Massey products, but we will not make use of it in this paper. Instead, we will use another, better suited obstruction to 1-formality, which is provided by the following theorem.

**Theorem 6.3** ([17]). *Let  $G$  be a finitely generated, 1-formal group. Then all irreducible components of  $\mathcal{R}_d^1(G)$  are rationally defined linear subspaces of  $H^1(G, \mathbb{C})$ , for all  $d \geq 0$ .*

**6.3. Formality properties of  $vP_n$  and  $vP_n^+$ .** Recall that the pure virtual braid groups  $vP_n$  and  $vP_n^+$  are graded-formal for all  $n \geq 1$ . Furthermore,  $vP_2^+ = \mathbb{Z}$  and  $vP_2 = F_2$ , and so both are 1-formal groups.

**Lemma 6.4.** *The groups  $vP_3^+$  and  $vP_3$  are both 1-formal.*

*Proof.* As shown in [17] (see also [42]), the free product of two 1-formal groups is 1-formal. Hence  $vP_3^+ \cong \mathbb{Z}^2 * \mathbb{Z}$  is also 1-formal.

Since the pure braid group  $P_4 \cong \overline{P}_4 \times \mathbb{Z}$  is 1-formal, Theorem 6.2 ensures that the subgroup  $\overline{P}_4$  is also 1-formal. On the other hand, we know from Lemma 2.4 that  $vP_3 \cong \overline{P}_4 * \mathbb{Z}$ . Thus, by Theorem 6.2, the group  $vP_3$  is 1-formal.  $\square$

**Lemma 6.5.** *The group  $vP_4^+$  is not 1-formal.*

*Proof.* As shown in Lemma 4.6, the resonance variety  $\mathcal{R}_1^1(vP_4^+)$  is an irreducible subvariety of  $H^1(vP_4^+, \mathbb{C})$ . Thus, this variety doesn't decompose into a finite union of linear subspaces, and so, by Theorem 6.3, the group  $vP_4^+$  is not 1-formal.  $\square$

**Theorem 6.6.** *The groups  $vP_n^+$  and  $vP_n$  are not 1-formal for  $n \geq 4$ .*

*Proof.* As shown in Lemma 2.3, there is a split injection from  $vP_n^+$  to  $vP_{n+1}^+$ . Since  $vP_4^+$  is not 1-formal, Theorem 6.2, then, insures that the groups  $vP_n^+$  are not 1-formal for  $n \geq 4$ .

By Corollary 2.2, there is a split monomorphism  $vP_n^+ \rightarrow vP_n$ . From Theorem 6.6, we know that the group  $vP_n^+$  is not 1-formal for  $n \geq 4$ . Therefore, by Theorem 6.2, the group  $vP_n$  is not 1-formal for  $n \geq 4$ .  $\square$

**Corollary 6.7.** *The groups  $vP_n^+$  and  $vP_n$  are not filtered formal for  $n \geq 4$ .*

To summarize, the groups  $vP_n$  and  $vP_n^+$  are always graded-formal. Furthermore, they are 1-formal (equivalently, filtered-formal) if and only if  $n \leq 3$ . This completes the proof of Theorem 1.2 from the Introduction.

## 7. CHEN LIE ALGEBRAS AND ALEXANDER INVARIANTS

In this section, we discuss the relationship between the Chen Lie algebra and the Alexander invariant of a finitely generated group.

**7.1. Chen Lie algebras and Chen ranks.** Let  $G$  be a finitely generated group. The *Chen Lie algebra* of  $G$ , as defined by Chen [10], is the associated graded Lie algebra of its second derived quotient,  $G/G''$ . The projection  $\pi: G \twoheadrightarrow G/G''$  induces an epimorphism,  $\text{gr}(\pi): \text{gr}(G) \twoheadrightarrow \text{gr}(G/G'')$ . It is readily verified that  $\text{gr}_k(\pi)$  is an isomorphism for  $k \leq 3$ .

The integers  $\theta_k(G) := \text{rank}(\text{gr}_k(G/G''))$  are called the *Chen ranks* of  $G$ .

**Lemma 7.1.** *The Chen Lie algebra of the product of two groups  $G_1$  and  $G_2$  is isomorphic to the direct sum  $\text{gr}(G_1/G_1'') \oplus \text{gr}(G_2/G_2'')$ , as graded Lie algebras.*

*Proof.* The canonical projections  $G_1 \times G_2 \rightarrow G_i$  for  $i = 1, 2$  restrict to homomorphisms on the second derived subgroups,  $(G_1 \times G_2)'' \rightarrow G_i''$ . Hence, there is an epimorphism  $\phi: G_1 \times G_2 / (G_1 \times G_2)'' \rightarrow G_1/G_1'' \times G_2/G_2''$ , inducing an epimorphism

$$(37) \quad \text{gr}(\phi): \text{gr}((G_1 \times G_2) / (G_1 \times G_2)'') \longrightarrow \text{gr}(G_1/G_1'') \oplus \text{gr}(G_2/G_2'').$$

By [12, Corollary 1.10], we have that

$$(38) \quad \theta_k(G_1 \times G_2) = \theta_k(G_1) + \theta_k(G_2).$$

Hence, the homomorphism  $\text{gr}(\phi)$  is an isomorphism of graded Lie algebras.  $\square$

In [42, Theorem 8.4], we prove the following.

**Theorem 7.2 ([42]).** *Let  $G$  be a finitely generated group. The canonical projection  $G \twoheadrightarrow G/G''$  induces then an epimorphism of graded Lie algebras,*

$$(39) \quad \Phi: \text{gr}(G) / \text{gr}(G)'' \longrightarrow \text{gr}(G/G'').$$

*Furthermore, if  $G$  is filtered-formal, then the above map is an isomorphism.*

In the case when  $G$  is 1-formal, this theorem recovers a result from [33], which insures there is a graded Lie algebra isomorphism  $\mathfrak{h}(G) / \mathfrak{h}(G)'' \cong \text{gr}(G/G'')$ .

**7.2. Alexander invariants.** Once again, let  $G$  be a finitely generated group. Let us consider the  $\mathbb{C}$ -vector space  $H_1(G', \mathbb{C}) = G'/G'' \otimes \mathbb{C}$ . This vector space can be viewed as a (finitely generated) module over the group algebra  $\mathbb{C}[H]$ , with the abelianization  $H = G/G'$  acting on  $G'/G''$  by conjugation. Following [30], we denote this module by  $B_{\mathbb{C}}(G)$ , or  $B(G)$  for short, and call it the *Alexander invariant* of  $G$ . We refer to [30, 12, 33] for ways to compute presentations for the module  $B(G)$  in various degrees of generality.

The module  $B = B(G)$  may be filtered by powers of the augmentation ideal,  $I = \ker(\varepsilon: \mathbb{C}[H] \rightarrow \mathbb{C})$ , where  $\varepsilon$  is the ring map defined by  $\varepsilon(h) = 1$  for all  $h \in H$ . The associated graded module,

$$(40) \quad \text{gr}(B) = \bigoplus_{k \geq 0} I^k B / I^{k+1} B,$$

then, is a module over the graded ring  $\text{gr}(\mathbb{C}[H]) = \bigoplus_{k \geq 0} I^k / I^{k+1}$ . We call this module the *associated graded Alexander invariant* of  $G$ .

Work of W. Massey [30] implies that the map  $j: G'/G'' \rightarrow G/G''$  restricts to isomorphisms

$$(41) \quad I^k B \longrightarrow \Gamma_{k+2}(G/G'')$$

for all  $k \geq 0$ . Taking successive quotients of the respective filtrations and tensoring with  $\mathbb{C}$ , we obtain isomorphisms

$$(42) \quad \text{gr}_k(j): \text{gr}_k(B(G)) \longrightarrow \text{gr}_{k+2}(G/G'') \quad \text{for } k \geq 0.$$

Consequently, the Chen ranks of  $G$  can be expressed in terms of the Hilbert series of the graded module  $\text{gr}(B(G))$ , as follows:

$$(43) \quad \sum_{k \geq 2} \theta_k(G) \cdot t^{k-2} = \text{Hilb}(\text{gr}(B(G)), t).$$

**Remark 7.3.** If the group  $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$  is a finitely presented, commutator-relators group, then the Hilbert series of the module  $\text{gr}(B(G))$  may be computed using the algorithm from [11, 13]. To start with, identify  $\mathbb{C}[H]$  with  $\Lambda = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ . The Alexander invariant of  $G$  admits then a finite presentation for the form

$$(44) \quad \Lambda^{\binom{n}{3}} \oplus \Lambda^m \xrightarrow{\delta_3 + \nu_G} \Lambda^{\binom{n}{2}} \longrightarrow B(G) \longrightarrow 0,$$

Here,  $\delta_i$  is the  $i$ -th differential in the standard Koszul resolution of  $\mathbb{C}$  over  $\Lambda$ , and  $\nu_G$  is a map satisfying  $\delta_2 \circ \nu_G = D_G$ , where  $D_G$  is the abelianization of the Jacobian matrix of Fox derivatives of the relators. Next, one computes a Gröbner basis for the module  $B(G)$ , in a suitable monomial ordering. An application of the standard tangent cone algorithm yields then a presentation for  $\text{gr}(B(G))$ , from which ones computes the Hilbert series of  $\text{gr}(B(G))$ . Finally, the Chen ranks of  $G$  are given by formula (43).

**7.3. Chen ranks of the free groups.** As shown in [10], the Chen ranks of the free group  $F_n$  are given by  $\theta_1(F_n) = n$  and

$$(45) \quad \theta_k(F_n) = \binom{n+k-2}{k} (k-1) \text{ for } k \geq 2.$$

Equivalently, by Massey's formula (43), the Hilbert series for the associated graded Alexander invariant of  $F_n$  is given by

$$(46) \quad \text{Hilb}(\text{gr}(B(F_n)), t) = \frac{1}{t^2} \cdot \left( 1 - \frac{1-nt}{(1-t)^n} \right),$$

an identity which can also be verified directly, by using the fact that  $B(F_n)$  is the cokernel of the third boundary map of the Koszul resolution  $\bigwedge(\mathbb{Z}^n) \otimes \mathbb{C}[\mathbb{Z}^n]$ .

From formula (46), we see that the generating and exponential generating functions for the Hilbert series of the associated graded Alexander invariants of the sequence of free groups  $\mathbf{F} = \{F_n\}_{n \geq 1}$  are given by

$$(47) \quad \begin{aligned} \sum_{n=1}^{\infty} \text{Hilb}(\text{gr}(B(F_n)), t) \cdot u^n &= \frac{u^2}{(1-u)(1-t-u)^2}, \\ \sum_{n=1}^{\infty} \text{Hilb}(\text{gr}(B(F_n)), t) \cdot \frac{u^n}{n!} &= \frac{e^u}{t^2} + \frac{e^{u/(1-t)}}{t^2} \left( \frac{tu}{1-t} - 1 \right). \end{aligned}$$

**7.4. Chen ranks of the pure braid groups.** A comprehensive algorithm for computing the Chen ranks of finitely presented groups was developed in [11, 13], leading to the following expressions for the Chen ranks of the pure braid groups:

$$(48) \quad \theta_1(P_n) = \binom{n}{2}, \quad \theta_2(P_n) = \binom{n}{3}, \quad \text{and} \quad \theta_k(P_n) = (k-1) \binom{n+1}{4} \text{ for } k \geq 3,$$

or, equivalently,

$$(49) \quad \text{Hilb}(\text{gr}(B(P_n)), t) = \binom{n+1}{4} \frac{1}{(1-t)^2} - \binom{n}{4}.$$

It follows that the two generating functions for the Hilbert series of the associated graded Alexander invariants of the sequence of pure braid groups  $\mathbf{P} = \{P_n\}_{n \geq 1}$  are given by

$$(50) \quad \begin{aligned} \sum_{n=1}^{\infty} \text{Hilb}(\text{gr}(B(P_n)), t) \cdot u^n &= \frac{u^3}{(1-u)^5} \left( \frac{1}{(1-t)^2} - u \right), \\ \sum_{n=1}^{\infty} \text{Hilb}(\text{gr}(B(P_n)), t) \cdot \frac{u^n}{n!} &= \frac{e^u u^3}{24} \left( \frac{u+4}{(1-t)^2} - u \right). \end{aligned}$$

## 8. INFINITESIMAL ALEXANDER INVARIANTS

We now turn to the infinitesimal Alexander invariants,  $\mathfrak{B}(\mathfrak{h}(G))$  and  $\mathfrak{B}(\text{gr}(G))$ , and the way they relate to the associated graded Alexander invariant,  $\text{gr}(B(G))$ .

**8.1. The infinitesimal Alexander invariant of a Lie algebra.** We start in a more general context. Let  $\mathfrak{g}$  be a finitely generated graded Lie algebra, with graded pieces  $\mathfrak{g}_k$ , for  $k \geq 1$ . Then both the derived algebra,  $\mathfrak{g}'$ , and the second derived algebra  $\mathfrak{g}'' = (\mathfrak{g}')'$ , are graded sub-Lie algebras. Thus, the maximal metabelian quotient,  $\mathfrak{g}/\mathfrak{g}''$ , is in a natural way a graded Lie algebra, with derived subalgebra  $\mathfrak{g}'/\mathfrak{g}''$ . Define the Chen ranks of  $\mathfrak{g}$  to be

$$(51) \quad \theta_k(\mathfrak{g}) = \dim(\mathfrak{g}/\mathfrak{g}'')_k.$$

Following [33], we associate to  $\mathfrak{g}$  a graded module over the symmetric algebra  $S = \text{Sym}(\mathfrak{g}_1)$ , as follows. The adjoint representation of  $\mathfrak{g}_1$  on  $\mathfrak{g}/\mathfrak{g}''$  defines an  $S$ -action on  $\mathfrak{g}'/\mathfrak{g}''$ , given by  $h \cdot \bar{x} = \overline{[h, x]}$ , for  $h \in \mathfrak{g}_1$  and  $x \in \mathfrak{g}'$ . Clearly, this action is compatible with the grading on  $\mathfrak{g}'/\mathfrak{g}''$ . We then let the *infinitesimal Alexander invariant* of  $\mathfrak{g}$  to be the graded  $S$ -module

$$(52) \quad \mathfrak{B}(\mathfrak{g}) = \mathfrak{g}'/\mathfrak{g}''.$$

Assume now that the graded Lie algebra  $\mathfrak{g} = \bigoplus_{k \geq 1} \mathfrak{g}_k$  is generated in degree 1. We then have  $\mathfrak{g}' = \bigoplus_{k \geq 2} \mathfrak{g}_k$ . Thus, since the grading for  $S$  starts with  $S_0 = \mathbb{C}$ , we are led to define the grading on  $\mathfrak{B}(\mathfrak{g})$  as

$$(53) \quad \mathfrak{B}(\mathfrak{g})_k = (\mathfrak{g}'/\mathfrak{g}'')_{k+2}, \text{ for } k \geq 0.$$

We then have the following ‘infinitesimal’ version of Massey’s formula (43).

**Proposition 8.1.** *Let  $\mathfrak{g}$  be a finitely generated, graded Lie algebra  $\mathfrak{g}$  generated in degree 1. Then the Chen ranks of  $\mathfrak{g}$  are given by*

$$(54) \quad \sum_{k \geq 2} \theta_k(\mathfrak{g}) \cdot t^{k-2} = \text{Hilb}(\mathfrak{B}(\mathfrak{g}), t).$$

*Proof.* Since  $\mathfrak{g}$  is generated in degree 1, we have that  $\mathfrak{g}/\mathfrak{g}' \cong \mathfrak{g}_1$ . Using now the exact sequence of graded Lie algebras  $0 \rightarrow \mathfrak{g}'/\mathfrak{g}'' \rightarrow \mathfrak{g}/\mathfrak{g}'' \rightarrow \mathfrak{g}/\mathfrak{g}' \rightarrow 0$ , we find that  $(\mathfrak{g}/\mathfrak{g}'')_k = (\mathfrak{g}'/\mathfrak{g}'')_k$  for all  $k \geq 2$ . The claim then follows from (51) and (53).  $\square$

**8.2. The infinitesimal Alexander invariants of a group.** Let again  $G$  be a finitely generated group. Denote by  $H = G_{ab}$  its abelianization, and identify  $\mathfrak{h}_1(G) = \text{gr}_1(G)$  with  $H \otimes \mathbb{C}$ . Finally, set  $S = \text{Sym}(H \otimes \mathbb{C})$ . The procedure outlined in §8.1 yields two  $S$ -modules attached to  $G$ .

The first one is  $\mathfrak{B}(G) = \mathfrak{B}(\mathfrak{h}(G))$ , the infinitesimal Alexander invariant of the holonomy Lie algebra of  $G$ . (When  $G$  is a finitely presented, commutator-relators group, this  $S$ -module coincides with the ‘linearized Alexander invariant’ from [12, 32], see [33]). The second one is  $\mathfrak{B}(\text{gr}(G))$ , the infinitesimal Alexander invariant of the associated graded Lie algebra of  $G$ . The next result provides a natural comparison map between these  $S$ -modules.

**Proposition 8.2.** *The canonical epimorphism  $\Psi: \mathfrak{h}(G) \twoheadrightarrow \text{gr}(G)$  from (27) induces an epimorphism of  $S$ -modules,*

$$\psi: \mathfrak{B}(\mathfrak{h}(G)) \longrightarrow \mathfrak{B}(\text{gr}(G)).$$

Moreover, if  $G$  is graded-formal, then  $\psi$  is an isomorphism.

*Proof.* The graded Lie algebra map  $\Psi: \mathfrak{h}(G) \rightarrow \text{gr}(G)$  preserves derived series, and thus induces an epimorphism  $\psi: \mathfrak{h}(G)'/\mathfrak{h}(G)'' \rightarrow \text{gr}(G)'/\text{gr}(G)''$ . By the discussion from §8.1, this map can also be viewed as a map  $\psi: \mathfrak{B}(\mathfrak{h}(G)) \rightarrow \mathfrak{B}(\text{gr}(G))$  of graded  $S$ -modules.

Finally, if  $G$  is graded-formal, i.e., if  $\Psi$  is an isomorphism, then clearly  $\psi$  is also an isomorphism.  $\square$

**Remark 8.3.** For a finitely presented group  $G$ , a finite presentation for the  $S$ -module  $\mathfrak{B}(\mathfrak{h}(G))$  is given in [33]. This presentation may be used to compute the holonomy Chen ranks  $\theta_k(\mathfrak{h}(G))$  from the Hilbert series of  $\mathfrak{B}(\mathfrak{h}(G))$ , using an approach analogous to the one described in Remark 7.3. We refer to [44] for more information on this subject, illustrated with detailed computations for the (upper) pure welded braid groups.

**8.3. Another filtration on  $G'/G''$ .** Next, we compare the module  $\mathfrak{B}(\text{gr}(G))$  to another, naturally defined  $S$ -module associated to the group  $G$ . Let  $\text{gr}^{\tilde{\Gamma}}(G'/G'')$  be the associated graded Lie algebra of  $G'/G''$  with respect to the induced filtration

$$(55) \quad \tilde{\Gamma}_k(G'/G'') := (G'/G'') \cap \Gamma_k(G/G'').$$

The terms of this filtration fit into short exact sequences

$$(56) \quad 0 \longrightarrow \tilde{\Gamma}_k(G'/G'') \longrightarrow \Gamma_k(G/G'') \longrightarrow \Gamma_k(G/G') \longrightarrow 0.$$

Noting that  $\Gamma_k(G/G') = 0$  for  $k \geq 2$ , we deduce that

$$(57) \quad \tilde{\Gamma}_k(G'/G'') = \Gamma_k(G/G''), \text{ for } k \geq 2.$$

As before, it is readily checked that the adjoint representation of  $\text{gr}_1(G/G'') = H \otimes \mathbb{C}$  on  $\text{gr}(G/G'')$  induces an  $S$ -action on  $\text{gr}^{\tilde{\Gamma}}(G'/G'')$ , preserving the grading. Hence, the Lie algebra  $\mathfrak{C}(G) := \text{gr}^{\tilde{\Gamma}}(G'/G'')$  can also be viewed as a graded module over  $S$ , by setting

$$(58) \quad \mathfrak{C}(G)_k = \text{gr}_{k+2}^{\tilde{\Gamma}}(G'/G'').$$

**Proposition 8.4.** *The canonical morphism of graded Lie algebras  $\Phi: \text{gr}(G)/\text{gr}(G)'' \rightarrow \text{gr}(G/G'')$  from (39) induces an epimorphism of  $S$ -modules,*

$$\varphi: \mathfrak{B}(\text{gr}(G)) \twoheadrightarrow \mathfrak{C}(G).$$

Moreover, if  $G$  is filtered-formal, then  $\varphi$  is an isomorphism.

*Proof.* The map  $\Phi$  fits into the following commutative diagram of graded Lie algebras,

$$(59) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{gr}(G)'/\text{gr}(G)'' & \longrightarrow & \text{gr}(G)/\text{gr}(G)'' & \longrightarrow & \text{gr}(G)/\text{gr}(G)' \longrightarrow 0 \\ & & \downarrow \varphi & & \downarrow \Phi & & \downarrow \text{id} \\ 0 & \longrightarrow & \text{gr}^{\tilde{\Gamma}}(G'/G'') & \xrightarrow{\text{gr}(j)} & \text{gr}(G/G'') & \longrightarrow & \text{gr}(G/G') \longrightarrow 0. \end{array}$$

Thus,  $\Phi$  induces a morphism of graded Lie algebras,  $\varphi: \text{gr}(G)'/\text{gr}(G)'' \rightarrow \text{gr}^{\tilde{\Gamma}}(G'/G'')$ , as indicated above. By the Five Lemma,  $\varphi$  is surjective. Observe that  $\text{gr}(G)/\text{gr}(G)' \cong$

$\text{gr}(G/G') \cong H \otimes \mathbb{C}$  acts on both the source and target of  $\varphi$  by adjoint representations. Hence, upon regrading according to (53) and (58), the map  $\varphi: \mathfrak{B}(\text{gr}(G)) \rightarrow \mathfrak{C}(G)$  becomes a morphism of  $S$ -modules.

If  $G$  is filtered-formal, then, according to Theorem 7.2, the map  $\Phi$  is an isomorphism of graded Lie algebras. Hence, the induced map  $\varphi$  is an isomorphism of  $S$ -modules.  $\square$

**8.4. Comparison with the associated graded Alexander invariant.** Finally, we identify the associated graded Alexander invariant  $\text{gr}(B(G))$  with the  $S$ -module  $\mathfrak{C}(G)$  defined above. To do that, we first identify the respective ground rings.

Choose a basis  $\{x_1, \dots, x_n\}$  for the torsion-free part of  $H = G_{\text{ab}}$ . We may then identify the group algebra  $\text{gr}(\mathbb{C}[H])$  with the polynomial algebra  $R = \mathbb{C}[s_1, \dots, s_n]$ , where  $s_i$  corresponds to  $\overline{x_i - 1} \in I/I^2$ , see Quillen [37]. On the other hand, we may also identify the symmetric algebra  $S = \text{Sym}(H \otimes \mathbb{C})$  with the polynomial algebra  $\mathbb{C}[x_1, \dots, x_n]$ . The desired ring isomorphism,  $R \cong S$ , is gotten by sending  $s_i$  to  $x_i$ .

**Proposition 8.5.** *Under the above identification  $R \cong S$ , the graded  $R$ -module  $\text{gr}(B(G))$  is canonically isomorphic to the graded  $S$ -module  $\mathfrak{C}(G)$ .*

*Proof.* Recall from §7.2 that the inclusion map  $j: G'/G'' \rightarrow G/G''$  restricts to an isomorphism  $I^k B(G) \rightarrow \Gamma_{k+2}(G/G'')$  for each  $k \geq 0$ . Using the induced filtration  $\tilde{\Gamma}$  from (55) and the identification (57), we obtain  $\mathbb{C}$ -linear isomorphisms  $I^k B(G) \cong \tilde{\Gamma}_{k+2}(G'/G'')$ , for all  $k \geq 0$ . Taking the successive quotients of the respective filtrations and regrading according to (58), we obtain a  $\mathbb{C}$ -linear isomorphism  $\text{gr}(B(G)) \cong \mathfrak{C}(G)$ .

Under the identification  $\text{gr}(\mathbb{C}[H]) \cong R$ , the associated graded Alexander invariant of  $G$  may be viewed as a graded  $R$ -module, with  $R$ -action defined by

$$(60) \quad s_i(\overline{z}) = (\overline{x_i - 1})\overline{z} = \overline{x_i z x_i^{-1} - z} = \overline{[x_i, z]z - z} = \overline{[x_i, z]} + \overline{z} - \overline{z} = \overline{[x_i, z]},$$

for all  $z \in G'$ . (In this computation, we follow the convention from [30], and view the Alexander invariant  $B(G) = G'/G''$  as an additive group; however, when we consider the induced filtration  $\tilde{\Gamma}$  on  $G'/G''$ , we view it as a multiplicative subgroup of  $G/G''$ .)

Finally, recall that  $\mathfrak{C}_\bullet(G) = \text{gr}_{\bullet-2}^{\tilde{\Gamma}}(G'/G'')$  is an  $S$ -module, with  $S$ -action given by  $x_i(\overline{z}) = \overline{[x_i, z]}$ . Hence, the aforementioned isomorphism  $R \cong S$  identifies the  $R$ -module  $\text{gr}(B(G))$  with the  $S$ -module  $\mathfrak{C}(G)$ .  $\square$

**8.5. Discussion.** In the 1-formal case, we obtain the following corollary, which can also be deduced from [17, Theorem 5.6].

**Corollary 8.6.** *Let  $G$  be a 1-formal group. Then  $\text{gr}(B(G)) \cong \mathfrak{B}(G)$ , as modules over the polynomial ring  $S = \text{gr}(\mathbb{C}[H])$ .*

*Proof.* Follows at once from Propositions 8.2, 8.4, and 8.5.  $\square$

Using those propositions once again, we obtain another corollary.

**Corollary 8.7.** *Let  $G$  be a finitely generated group. The following then hold.*

- (1)  $\theta_k(\text{gr}(G)) \leq \theta_k(\mathfrak{h}(G))$ , with equality if  $k \leq 2$ , or if  $G$  is graded-formal.

(2)  $\theta_k(G) \leq \theta_k(\text{gr}(G))$ , with equality if  $k \leq 3$ , or if  $G$  is filtered-formal.

The graded-formality assumption from part (1) of the above corollary is clearly necessary for the equality  $\theta_k(\text{gr}(G)) = \theta_k(\mathfrak{h}(G))$  to hold for all  $k$ . On the other hand, it is not clear whether the filtered-formality hypothesis from part (2) is necessary for the equality  $\theta_k(G) = \theta_k(\text{gr}(G))$  to hold in general. In view of several computations (some of which are summarized in the next section), we are led to formulate the following question.

**Question 8.8.** Suppose  $G$  is a graded-formal group. Does the equality  $\theta_k(G) = \theta_k(\text{gr}(G))$  hold for all  $k$ ?

## 9. CHEN RANKS AND RESONANCE

In this section, we detect the relationship between the Chen ranks and the resonance varieties. We also compute the Chen ranks of some (upper) pure virtual braid groups.

**9.1. The Chen ranks formula.** Let  $G$  be a finitely presented, commutator-relators group. As shown in [32], for each  $d \geq 1$ , the resonance variety  $\mathcal{R}_d^1(G)$  coincides, at least away from the origin  $0 \in H^1(G; \mathbb{C})$ , with the support variety of the annihilator of  $d$ -th exterior power of the infinitesimal Alexander invariant; that is,

$$(61) \quad \mathcal{R}_d^1(G) = V\left(\text{Ann}\left(\bigwedge^d \mathfrak{B}(\mathfrak{h}(G))\right)\right).$$

The first author conjectured in [40] that for  $k \gg 0$ , the Chen ranks of an arrangement group  $G$  are given by the *Chen ranks formula*

$$(62) \quad \theta_k(G) = \sum_{m \geq 2} h_m(G) \cdot \theta_k(F_m),$$

where  $h_m(G)$  is the number of  $m$ -dimensional irreducible components of  $\mathcal{R}_1^1(G)$ . A positive answer to this conjecture was given in [14] for a class of 1-formal groups which includes arrangement groups. To state this result, recall that a subspace  $U \subset H^1(G; \mathbb{C})$  is called *isotropic* if the cup product  $U \wedge U \rightarrow H^2(G; \mathbb{C})$  is the zero map.

**Theorem 9.1** ([14]). *Let  $G$  be a finitely presented, commutator-relators 1-formal group. Assume that the components of  $\mathcal{R}_1^1(G)$  are isotropic, projectively disjoint, and reduced as a scheme. Then the Chen ranks formula holds for  $G$ .*

Using Hilbert series of the Alexander invariants, the Chen ranks formula (62) translates into the equivalent statement that

$$(63) \quad \text{Hilb}(\text{gr}(B(G)), t) - \sum_{m \geq 2} h_m(G) \cdot \text{Hilb}(\text{gr}(B(F_m)), t)$$

is a polynomial (in the variable  $t$ ).

**Example 9.2.** The pure braid group  $P_n$  is an arrangement group, and thus satisfies the hypothesis of Theorem 9.1. In fact, we know from Proposition 4.4 that the resonance variety  $\mathcal{R}_1^1(P_n)$  has  $\binom{n}{3} + \binom{n}{4} = \binom{n+1}{4}$  irreducible components, all of dimension 2. Thus, the computation from (48) agrees with the one predicted by formula (62), for all  $k \geq 3$ .

As noted in [14], it is easy to find examples of non-1-formal groups for which the Chen ranks formula does not hold. For instance, if  $G = F_2/\Gamma_3(F_2)$  is the Heisenberg group, then  $\mathcal{R}_1^1(G) = H^1(G, \mathbb{C}) = \mathbb{C}^2$ , and thus formula (62) would predict in this case that  $\theta_k(G) = \theta_k(F_2)$  for  $k$  large enough, where in reality  $\theta_k(G) = 0$  for  $k \geq 3$ . On the other hand, here is an example of a finitely presented, commutator-relators group which satisfies the Chen ranks formula, yet which is *not* 1-formal.

**Example 9.3.** Using the notation from [32], let  $\mathcal{A} = \mathcal{A}(31425)$  be the ‘horizontal’ arrangement of 2-planes in  $\mathbb{R}^4$  determined by the specified permutation, and let  $G$  be the fundamental group of its complement. From [32, Example 6.5], we know that  $\mathcal{R}_1^1(G)$  is an irreducible cubic hypersurface in  $H^1(G, \mathbb{C}) = \mathbb{C}^5$ . Hence, by Theorem 6.3, the group  $G$  is not 1-formal (for an alternative argument, see [17, Example 8.2]). On the other hand, the singularity link determined by  $\mathcal{A}$  has all linking numbers  $\pm 1$ , and thus satisfies the Murasugi Conjecture, that is,  $\theta_k(G) = \theta_k(F_4)$ , for all  $k \geq 2$ , see [31, 33]. Therefore, the Chen ranks formula holds for the group  $G$ .

**9.2. Products and coproducts.** We now analyze the way the Chen ranks formula (62) behaves under (finite) products and coproducts of groups.

**Lemma 9.4.** *Let  $G_1$  and  $G_2$  be two finitely generated groups. The number of  $m$ -dimensional irreducible components of the corresponding first resonance varieties satisfies the following additivity formula,*

$$(64) \quad h_m(G_1 \times G_2) = h_m(G_1) + h_m(G_2).$$

*Proof.* We start by identifying the affine space  $H^1(G_1 \times G_2; \mathbb{C})$  with  $H^1(G_1; \mathbb{C}) \times H^1(G_2; \mathbb{C})$ . Next, by Proposition 4.2, we have that

$$(65) \quad \mathcal{R}_1^1(G_1 \times G_2) = \mathcal{R}_1^1(G_1) \times \{0\} \cup \{0\} \times \mathcal{R}_1^1(G_2).$$

Suppose  $\mathcal{R}_1^1(G_1) = \bigcup_{i=1}^s A_i$  and  $\mathcal{R}_1^1(G_2) = \bigcup_{j=1}^t B_j$  are the decompositions into irreducible components for the respective varieties. Then  $A_i \times \{0\}$  and  $\{0\} \times B_j$  are irreducible subvarieties of  $\mathcal{R}_1^1(G_1 \times G_2)$ . Observe now that  $\mathcal{R}_1^1(G_1) \times \{0\}$  and  $\{0\} \times \mathcal{R}_1^1(G_2)$  intersect only at 0. It follows that

$$(66) \quad \mathcal{R}_1^1(G_1 \times G_2) = \bigcup_{i=1}^s A_i \times \{0\} \cup \bigcup_{j=1}^t \{0\} \times B_j$$

is the irreducible decomposition for the first resonance variety of  $G_1 \times G_2$ . The claimed additivity formula follows.  $\square$

**Corollary 9.5.** *If both  $G_1$  and  $G_2$  satisfy the Chen ranks formula, then  $G_1 \times G_2$  also satisfies the Chen ranks formula.*

*Proof.* Follows at once from formulas (38) and (64).  $\square$

However, even if both  $G_1$  and  $G_2$  satisfy the Chen ranks formula, the free product  $G_1 * G_2$  may not satisfy this formula. We illustrate this phenomenon with an infinite family of examples.

**Example 9.6.** Let  $G_n = \mathbb{Z} * \mathbb{Z}^{n-1}$ . Clearly, both factors of this free product satisfy the Chen ranks formula; in fact, both factors satisfy the hypothesis of Theorem 9.1. Moreover,  $G_n$  is 1-formal and  $\mathcal{R}^1(G_n)$  is projectively disjoint and reduced as a scheme. Using Theorem 4.1(3) and Lemma 6.2 from [34], a short computation reveals that

$$(67) \quad \sum_{k \geq 2} \theta_k(G_n) t^k = t \frac{1 - (1-t)^{n-1}}{(1-t)^n}.$$

On the other hand, if  $n \geq 2$ , then  $\mathcal{R}_1^1(G_n) = H^1(G_n, \mathbb{C})$ , by Proposition 4.3. Thus, formula (62) would say that  $\theta_k(G_n) = \theta_k(F_n)$  for  $k \gg 0$ . However, comparing formulas (46) and (67), we find that

$$(68) \quad \theta_k(F_n) - \theta_k(G_n) = \sum_{i=2}^k \theta_i(F_{n-1}).$$

Hence, if  $n \geq 3$ , the group  $G_n$  does *not* satisfy the Chen ranks formula. Note that  $G_n$  also does not satisfy the isotropicity hypothesis of Theorem 9.1, since the restriction of the cup product to the factor  $\mathbb{Z}^{n-1}$  is nonzero, again provided that  $n \geq 3$ .

**9.3. Chen ranks of  $vP_3$  and  $vP_3^+$ .** We now return to the pure virtual braid groups, and study their Chen ranks. Recall that  $vP_2^+ = \mathbb{Z}$  and  $vP_2 = F_2$ , so we may as well assume  $n \geq 3$ . We start with the case  $n = 3$ .

**Proposition 9.7.** *The groups  $vP_3^+$  and  $vP_3$  do not satisfy the Chen ranks formula, despite the fact that they are both 1-formal, and their first resonance varieties are projectively disjoint and reduced as schemes.*

*Proof.* Recall that  $vP_3^+ \cong \mathbb{Z}^2 * \mathbb{Z}$ . Thus, the claim for  $vP_3^+$  is handled by the argument from Example 9.6.

Next, recall that  $vP_3 \cong \overline{P}_4 * \mathbb{Z}$ . We know from Lemma 6.4 that  $vP_3$  is 1-formal. Furthermore, we know from Proposition 4.5 that  $\mathcal{R}_1^1(vP_3) = H^1(vP_3, \mathbb{C})$ . Clearly, this variety is projectively disjoint and reduced as a scheme. Using the algorithm described in Remark 7.3, we find that the Hilbert series of the associated graded Alexander invariants of  $vP_3$  is given by

$$(69) \quad \text{Hilb}(\text{gr}(B(vP_3)), t) = (9 - 20t + 15t^2 - 4t^4 + t^5)/(1-t)^6.$$

On the other hand, as noted above,  $\mathcal{R}_1^1(vP_3) = \mathbb{C}^6$ . Using (46) and (69), we compute

$$(70) \quad \text{Hilb}(\text{gr}(B(vP_3)), t) - \text{Hilb}(\text{gr}(B(F_6)), t) = (-6 + 6t^3 - 5t^4 + t^5)/(1-t)^6.$$

Since this expression is *not* a polynomial in  $t$ , we conclude that formula (62) does not hold for  $vP_3$ , and this ends the proof.  $\square$

**Remark 9.8.** The resonance varieties of  $vP_3^+$  and  $vP_3$  are not isotropic, since both groups have non-vanishing cup products stemming from the subgroups  $\mathbb{Z}^2$  and  $\overline{P}_4$ , respectively. Thus, once again, the groups  $vP_3^+$  and  $vP_3$  illustrate the necessity of the isotropicity hypothesis from Theorem 9.1.

**9.4. Holonomy Chen ranks of  $vP_n$  and  $vP_n^+$ .** We conclude with a summary of what else we know about the Chen ranks of the pure virtual braid groups, as well as the Chen ranks of the respective holonomy Lie algebras and associated graded Lie algebras.

**Proposition 9.9.** *The following equalities of Chen ranks hold.*

- (1)  $\theta_k(\mathfrak{h}(vP_n^+)) = \theta_k(\text{gr}(vP_n^+))$  and  $\theta_k(\mathfrak{h}(vP_n)) = \theta_k(\text{gr}(vP_n))$ , for all  $n$  and  $k$ .
- (2)  $\theta_k(\mathfrak{h}(vP_n^+)) = \theta_k(vP_n^+)$  for  $n \leq 6$  and all  $k$ .
- (3)  $\theta_k(\mathfrak{h}(vP_n)) = \theta_k(vP_n)$  for  $n \leq 3$  and all  $k$ .

*Proof.* (1) Recall from Theorem 5.6 that the pure virtual braid groups  $vP_n$  and  $vP_n^+$  are graded-formal. Therefore, by Corollary 8.7, claim (2) holds.

For  $n \leq 3$ , claims (2) and (3) follow from the 1-formality of the groups  $vP_n^+$  and  $vP_n$  in that range, and Corollary 8.7.

Using now the algorithms described in Remarks 7.3 and 8.3, a Macaulay2 [28] computation reveals that

$$(71) \quad \begin{aligned} \sum_{k \geq 2} \theta_k(vP_4^+) t^{k-2} &= \sum_{k \geq 2} \theta_k(\mathfrak{h}(vP_4^+)) t^{k-2} = (8 - 3t + t^2)/(1-t)^4, \\ \sum_{k \geq 2} \theta_k(vP_5^+) t^{k-2} &= \sum_{k \geq 2} \theta_k(\mathfrak{h}(vP_5^+)) t^{k-2} = (20 + 15t + 5t^2)/(1-t)^4, \\ \sum_{k \geq 2} \theta_k(vP_6^+) t^{k-2} &= \sum_{k \geq 2} \theta_k(\mathfrak{h}(vP_6^+)) t^{k-2} = (40 + 35t - 40t^2 - 20t^3)/(1-t)^5. \end{aligned}$$

This establishes claim (2) for  $4 \leq n \leq 6$ , thereby completing the proof.  $\square$

It would be interesting to decide whether the equalities in parts (2) and (3) of the above proposition hold for all  $n$  and all  $k$ .

Finally, let us address the validity of the Chen ranks formula (62) for the pure virtual braid groups on  $n \geq 4$  strings. We know from Lemma 4.6 that  $\mathcal{R}_1^1(vP_4^+)$  has a single irreducible component of dimension 4. A similar Macaulay2 [28] computation shows that  $\mathcal{R}_1^1(vP_5^+)$  has 15 irreducible components, all of dimension 4. Using now (71), it is readily seen that the Chen ranks formula does not hold for either  $vP_4^+$  or  $vP_5^+$ . Based on this evidence, and some further computations, we expect that the Chen ranks formula does not hold for any of the groups  $vP_n$  and  $vP_n^+$  with  $n \geq 4$ .

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