Torsion in the homology of Milnor fibers of hyperplane arrangements

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Abstract As is well-known, the homology groups of the complement of a complex hyperplane arrangement are torsion-free. Nevertheless, as we showed in a recent paper [2], the homology groups of the Milnor fiber of such an arrangement can have non-trivial integer torsion. We give here a brief account of the techniques that go into proving this result, outline some of its applications, and indicate some further questions that it brings to light.

Introduction

This talk reported on the main results of [2]. Here, we give an outline of our approach and a summary of our conclusions. Our main result gives a construction of a family of projective hypersurfaces for which the Milnor fiber has torsion in homology. The hypersurfaces we use are hyperplane arrangements, for which techniques are available to examine the homology of finite cyclic covers quite explicitly, by reducing to rank 1 local systems.

The parameter spaces for rank 1 local systems with non-vanishing homology are known as characteristic varieties. In the special case of complex hyperplane arrangement complements, the combinatorial theory of multinets largely elucidates their structure, at least in degree 1. We make use of an iterated parallel connection construction to build arrangements with suitable characteristic varieties, then vary the characteristic of the field of definition

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in order to construct finite cyclic covers with torsion in first homology. These covers include the Milnor fiber. We now give some detail about each step.

The Milnor fibration

A classical construction due to J. Milnor associates to every homogeneous polynomial $f \in \mathbb{C}[z_1, \dots, z_\ell]$ a fiber bundle, with base space $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, total space the complement in $\mathbb{C}^\ell$ to the hypersurface defined by f, and projection map $f\colon \mathbb{C}^\ell \setminus f^{-1}(0) \to \mathbb{C}^*$.

The Milnor fiber $F = f^{-1}(1)$ has the homotopy type of a finite, $(\ell-1)$-dimensional CW-complex, while the monodromy of the fibration, $h\colon F \to F$, is given by $h(z) = e^{2\pi i/n} z$, where n is the degree of f. If f has an isolated singularity at the origin, then F is homotopic to a bouquet of $(\ell-1)$-spheres, whose number can be determined by algebraic means. In general, though, it is a rather hard problem to compute the homology groups of the Milnor fiber, even in the case when f completely factors into distinct linear forms: that is, when the hypersurface $\{f = 0\}$ is a hyperplane arrangement.

Building on our previous work with D. Cohen [1], we show there exist projective hypersurfaces (indeed, hyperplane arrangements) whose complements have torsion-free homology, but whose Milnor fibers have torsion in homology. Our main result can be summarized as follows.

Theorem 1. *For every prime $p \geq 2$, there is a hyperplane arrangement whose Milnor fiber has non-trivial p-torsion in homology.*

This resolves a problem posed by Randell [7, Problem 7], who conjectured that Milnor fibers of hyperplane arrangements have torsion-free homology. Our examples also give a refined answer to a question posed by Dimca and Némethi [3, Question 3.10]: torsion in homology may appear even if the hypersurface is defined by a reduced equation. We note the following consequence:

Corollary 1. *There are hyperplane arrangements whose Milnor fibers do not have a minimal cell structure.*

This stands in contrast with arrangement complements, which always admit perfect Morse functions. Our method also allows us to compute the homomorphism induced in homology by the monodromy, with coefficients in a field of characteristic not dividing the order of the monodromy.

It should be noted that our approach produces only examples of arrangements $\mathcal{A}$ for which the Milnor fiber $F(\mathcal{A})$ has torsion in q-th homology, for some $q > 1$. This leaves open the following question.

Question 1. Is there an arrangement $\mathcal{A}$ such that $H_1(F(\mathcal{A}), \mathbb{Z})$ has non-zero torsion?

Since our methods rely on complete reducibility, it is also natural to ask: do there exist projective hypersurfaces of degree n for which the Milnor fiber has homology p-torsion, where p divides n? If so, is there a hyperplane arrangement with this property?

A much-studied question in the subject is whether the Betti numbers of the Milnor fiber of an arrangement $\mathcal{A}$ are determined by the intersection lattice, $L(\mathcal{A})$. While we do not address this question directly, our result raises a related, and arguably even more subtle problem.

Question 2. Is the torsion in the homology of the Milnor fiber of a hyperplane arrangement combinatorially determined?

As a preliminary question, one may also ask: can one predict the existence of torsion in the homology of $F(\mathcal{A})$ simply by looking at $L(\mathcal{A})$? As it turns out, under fairly general assumptions, the answer is yes: if $L(\mathcal{A})$ satisfies certain very precise conditions, then automatically $H_*(F(\mathcal{A}),\mathbb{Z})$ will have non-zero torsion, in a combinatorially determined degree.

Hyperplane arrangements

Let $\mathcal{A}$ be a (central) arrangement of n hyperplanes in $\mathbb{C}^\ell$, defined by a polynomial $Q(\mathcal{A}) = \prod_{H\in\mathcal{A}} f_H$, where each f_H is a linear form whose kernel is H. The starting point of our study is the well-known observation that the Milnor fiber of the arrangement, $F(\mathcal{A})$, is a cyclic, n-fold regular cover of the projectivized complement, $U(\mathcal{A})$; this cover is defined by the homomorphism $\delta\colon \pi_1(U(\mathcal{A})) \twoheadrightarrow \mathbb{Z}_n$, taking each meridian generator x_H to 1.

Now, if $\Bbbk$ is an algebraically closed field whose characteristic does not divide n, then $H_q(F(\mathcal{A}),\Bbbk)$ decomposes as a direct sum, $\bigoplus_\rho H_q(U(\mathcal{A}),\Bbbk_\rho)$, where the rank 1 local systems $\Bbbk_\rho$ are indexed by characters $\rho\colon \pi_1(U(\mathcal{A})) \to \Bbbk^*$ that factor through δ. Thus, if there is such a character ρ for which $H_q(U(\mathcal{A}),\Bbbk_\rho) \neq 0$, but there is no corresponding character in characteristic 0, then the group $H_q(F(\mathcal{A}),\mathbb{Z})$ will have non-trivial p-torsion.

To find such characters, we first consider multi-arrangements $(\mathcal{A}, m)$, with positive integer weights m_H attached to each hyperplane $H \in \mathcal{A}$. The corresponding Milnor fiber, $F(\mathcal{A}, m)$, is defined by the homomorphism $\delta_m\colon \pi_1(U(\mathcal{A})) \twoheadrightarrow \mathbb{Z}_N$, $x_H \mapsto m_H$, where N denotes the sum of the weights. Fix a prime p. Starting with an arrangement $\mathcal{A}$ supporting a suitable multinet, we find a deletion $\mathcal{A}' = \mathcal{A} \setminus \{H\}$, and a choice of multiplicities m' on $\mathcal{A}'$ such that $H_1(F(\mathcal{A}', m'),\mathbb{Z})$ has p-torsion. Finally, we construct a "polarized" arrangement $\mathcal{B} = \mathcal{A}' \| m'$, and show that $H_*(F(\mathcal{B}),\mathbb{Z})$ has p-torsion.

Characteristic varieties

Our arguments depend on properties of the jump loci of rank 1 local systems. The *characteristic varieties* of a connected, finite CW-complex X are the subvarieties $\mathcal{V}^q_d(X, \Bbbk)$ of the character group $\widehat{G} = \operatorname{Hom}(G, \Bbbk^*)$, consisting of those characters ρ for which $H_q(X, \Bbbk_\rho)$ had dimension at least d.

Suppose $X^\chi \to X$ is a regular cover, defined by an epimorphism $\chi\colon G \to A$ to a finite abelian group, and if $\Bbbk$ is an algebraically closed field of characteristic p, where $p \nmid |\mathcal{A}|$, then $\dim H_q(X^\chi, \Bbbk) = \sum_{d\geq 1} |\operatorname{im}(\widehat{\chi}_{\Bbbk}) \cap \mathcal{V}^q_d(X, \Bbbk)|$, where $\widehat{\chi}_{\Bbbk}\colon \widehat{A} \to \widehat{G}$ is the induced morphism between character groups.

Theorem 2. *Let $X^\chi \to X$ be a regular, finite cyclic cover. Suppose that $\operatorname{im}(\widehat{\chi}_{\mathbb{C}}) \not\subseteq \mathcal{V}^q_1(X, \mathbb{C})$, but $\operatorname{im}(\widehat{\chi}_{\Bbbk}) \subseteq \mathcal{V}^q_1(X, \Bbbk)$, for some field $\Bbbk$ of characteristic p not dividing the order of the cover. Then $H_q(X^\chi, \mathbb{Z})$ has non-zero p-torsion.*

Multinets

In the case when $X = M(\mathcal{A})$ is the complement of a hyperplane arrangement, the positive-dimensional components of the characteristic variety $\mathcal{V}^1_1(X, \mathbb{C})$ have a combinatorial description, for which we refer in particular to work of Falk and Yuzvinsky in [5].

A *multinet* consists of a partition of $\mathcal{A}$ into at least 3 subsets $\mathcal{A}_1, \dots, \mathcal{A}_k$, together with an assignment of multiplicities, $m\colon \mathcal{A} \to \mathbb{N}$, and a subset $\mathcal{X}$ of the rank 2 flats, such that any two hyperplanes from different parts intersect at a flat in $\mathcal{X}$, and several technical conditions are satisfied: for instance, the sum of the multiplicities over each part $\mathcal{A}_i$ is constant, and for each flat $Z \in \mathcal{X}$, the sum $n_Z := \sum_{H\in\mathcal{A}_i\colon H\supset Z} m_H$ is independent of i. Each multinet gives rise to an orbifold fibration $X \to \mathbb{P}^1 \setminus \{k \text{ points}\}$; in turn, such a map yields by pullback an irreducible component of $\mathcal{V}^1_1(X, \mathbb{C})$.

We say that a multinet on $\mathcal{A}$ is *pointed* if for some hyperplane H, we have $m_H > 1$ and $m_H \mid n_Z$ for each flat $Z \subset H$ in $\mathcal{X}$. We show that the complement of the deletion $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ supports an orbifold fibration with base $\mathbb{C}^*$ and at least one multiple fiber. Consequently, for any prime $p \mid m_H$, and any sufficiently large integer r not divisible by p, there exists a regular, r-fold cyclic cover $Y \to U(\mathcal{A}')$ such that $H_1(Y, \mathbb{Z})$ has p-torsion.

Furthermore, we also show that any finite cyclic cover of an arrangement complement is dominated by a Milnor fiber corresponding to a suitable choice of multiplicities. Putting things together, we obtain the following result.

Theorem 3. *Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity vector m. Let p be a prime dividing m_H. There is then a choice of multiplicity vector m' on the deletion $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ such that $H_1(F(\mathcal{A}', m'), \mathbb{Z})$ has non-zero p-torsion.*

For instance, if $\mathcal{A}$ is the reflection arrangement of type B_3, defined by the polynomial $Q = xyz(x^2 - y^2)(x^2 - z^2)(y^2 - z^2)$, then $\mathcal{A}$ satisfies the conditions of the theorem, for $m = (2,2,2,1,1,1,1,1,1)$ and $H = \{z = 0\}$. Choosing then multiplicities $m' = (2,1,3,3,2,2,1,1)$ on $\mathcal{A}'$ shows that $H_1(F(\mathcal{A}', m'), \mathbb{Z})$ has non-zero 2-torsion.

Similarly, for primes $p > 2$, we use the fact that the reflection arrangement of the full monomial complex reflection group, $\mathcal{A}(p,1,3)$, admits a pointed multinet. This yields p-torsion in the first homology of the Milnor fiber of a suitable multi-arrangement on the deletion.

Parallel connections and polarizations

The last step of our construction replaces multi-arrangements with simple arrangements. We add more hyperplanes and increase the rank by means of suitable iterated parallel connections. The complement of the parallel connection of two arrangements is diffeomorphic to the product of the respective complements, by a result of Falk and Proudfoot [4]. Then the characteristic varieties of the result are given by a formula due to Papadima and Suciu [6].

We organize the process by noting that parallel connection of matroids has an operad structure, and we analyze a special case which we call the *polarization* of a multi-arrangement $(\mathcal{A}, m)$. By analogy with a construction involving monomial ideals, we use parallel connection to attach to each hyperplane H the central arrangement of m_H lines in $\mathbb{C}^2$, to obtain a simple arrangement we denote by $\mathcal{A}\| m$. A crucial point here is the connection between the respective Milnor fibers: the pullback of the cover $F(\mathcal{A}\| m) \to U(\mathcal{A}\| m)$ along the canonical inclusion $U(\mathcal{A}) \to U(\mathcal{A}\| m)$ is equivalent to the cover $F(\mathcal{A}, m) \to U(\mathcal{A})$. Using this fact, we prove the following.

Theorem 4. *Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H. There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ such that the Milnor fiber of the polarization $\mathcal{A}'\| m'$ has p-torsion in homology, in degree $1 + |\{K \in \mathcal{A}' : m'_K \geq 3\}|$.*

For instance, if $\mathcal{A}'$ is the deleted B_3 arrangement as above, then choosing $m' = (8,1,3,3,5,5,1,1)$ produces an arrangement $\mathcal{B} = \mathcal{A}'\| m'$ of 27 hyperplanes in $\mathbb{C}^8$, such that $H_6(F(\mathcal{B}), \mathbb{Z})$ has 2-torsion of rank 108.

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