

QUASI-KÄHLER GROUPS, 3-MANIFOLD GROUPS, AND FORMALITY

ALEXANDRU DIMCA^{1,2}, STEFAN PAPADIMA^{1,3}, AND ALEXANDER I. SUCIU⁴

ABSTRACT. In this note, we address the following question: Which 1-formal groups occur as fundamental groups of both quasi-Kähler manifolds and closed, connected, orientable 3-manifolds. We classify all such groups, at the level of Malcev completions, and compute their coranks. Dropping the assumption on realizability by 3-manifolds, we show that the corank equals the isotropy index of the cup-product map in degree one. Finally, we examine the formality properties of smooth affine surfaces and quasi-homogeneous isolated surface singularities. In the latter case, we describe explicitly the positive-dimensional components of the first characteristic variety for the associated singularity link.

1. INTRODUCTION

1.1. Kähler groups and 3-manifold groups. The following question was raised by W. Goldman and S. Donaldson in 1989, and independently by A. Reznikov in 1993: Which 3-manifold groups are Kähler groups? Both kinds of groups have a geometric origin: the first kind arise as fundamental groups of compact, connected, orientable 3-dimensional manifolds (without boundary), while the second kind arise as fundamental groups of compact, connected, Kähler manifolds.

In [36], Reznikov obtained a deep restriction on certain groups lying at the intersection of these two naturally defined classes. In [14], two of us answered the question in full generality: those groups which occur as both 3-manifold groups and Kähler groups are precisely the finite subgroups of $SO(4)$.

In this note, we pursue the analogous problem, for the more general class of quasi-Kähler groups. Under a 1-formality assumption (satisfied by all Kähler groups), we classify the Malcev completions of 3-manifold groups that fall into this class.

1.2. Quasi-Kähler groups. A manifold X is called *quasi-Kähler* if $X = \overline{X} \setminus D$, where \overline{X} is a compact, connected Kähler manifold and D is a divisor with normal crossings. If a group G can be realized as the fundamental group of such a manifold X , we say G is

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a quasi-Kähler group—or, a quasi-projective group, if \overline{X} is actually a projective variety. A natural question arises: which 3-manifold groups are quasi-Kähler?

In [12], we gave a complete answer to this question, within a restricted class of 3-manifolds: those manifolds $M = M(\mathcal{A})$ which occur as the boundary of a regular neighborhood of an arrangement \mathcal{A} of lines in $\mathbb{C}\mathbb{P}^2$. Using an explicit formula for the Alexander polynomial of $M(\mathcal{A})$ from [7], we showed that $G = \pi_1(M(\mathcal{A}))$ is quasi-Kähler if and only if \mathcal{A} is either a pencil of $n + 1$ lines, in which case $M(\mathcal{A}) = \#^n S^1 \times S^2$, or a pencil of n lines together with an extra line in general position, in which case $M(\mathcal{A}) = S^1 \times \Sigma_{n-1}$, where Σ_g is the closed, orientable surface of genus g .

Yet there are many more 3-manifold groups which are quasi-Kähler. For instance, if M is the link of an isolated surface singularity with \mathbb{C}^* -action, then its fundamental group is quasi-projective, yet not necessarily of the above form. A concrete example is provided by the Heisenberg nilmanifold, which occurs as the Brieskorn manifold $\Sigma(2, 3, 6)$.

1.3. Formality. An important topological property of compact Kähler manifolds is their formality. Following D. Sullivan [40], we say that a connected, finite-type CW-complex X is *formal* if its minimal model is quasi-isomorphic to $(H^*(X, \mathbb{Q}), 0)$. Examples of formal spaces include rational homology spheres, rational cohomology tori, surfaces, compact Lie groups, and complements of complex hyperplane arrangements. On the other hand, the only nilmanifolds which are formal are tori. Formality is preserved under wedges and products of spaces, and connected sums of manifolds.

A finitely generated group G is said to be *1-formal* if its Malcev completion—in the sense of D. Quillen [35]—is a quadratic complete Lie algebra. If X is formal, then $\pi_1(X)$ is 1-formal, but in general the converse does not hold.

As shown by Deligne, Griffiths, Morgan, and Sullivan [8], every compact Kähler manifold is formal. Thus, all Kähler groups are 1-formal and quasi-Kähler. Yet the converse is far from being true. Indeed, work of Morgan [27] and Kohno [19] shows that fundamental groups of complements of projective hypersurfaces are 1-formal. For example, let F_n be the free group of rank $n > 0$; then $F_n = \pi_1(\mathbb{C}\mathbb{P}^1 \setminus \{n + 1 \text{ points}\})$ is a 1-formal, quasi-projective group, but definitely not a Kähler group.

1.4. Cohomology jumping loci. A crucial tool for us are the cohomology jumping loci associated to a finite-type CW-complex X . The *characteristic varieties* $V_d^i(X)$ are algebraic subvarieties of $\text{Hom}(\pi_1(X), \mathbb{C}^*)$, while the *resonance varieties* $R_d^i(X)$ are homogeneous subvarieties of $H^1(X, \mathbb{C})$. The former are the jump loci for the cohomology of X with rank one twisted complex coefficients, while the latter are the jump loci for the homology of the cochain complexes arising from multiplication by degree 1 classes in $H^*(X, \mathbb{C})$. The jumping loci of a group are defined in terms of the jumping loci of the corresponding classifying space.

In general, the characteristic varieties are rather complicated objects. If the group $G = \pi_1(X)$ is 1-formal, though, we showed in [13] that the analytic germ of $V_d^1(X)$ at 1 coincides with the analytic germ of $R_d^1(X)$ at 0, and thus may be recovered from the cup-product map $\cup_X: H^1(X, \mathbb{C}) \wedge H^1(X, \mathbb{C}) \rightarrow H^2(X, \mathbb{C})$.

Foundational results on the structure of the cohomology support loci for local systems on Kähler and quasi-Kähler manifolds were obtained by Beauville [3], Green–Lazarsfeld

[15], Simpson [38], and Arapura [1]. In particular, if X is a quasi-Kähler manifold, then each variety $V_1^1(X, \mathbb{C})$ is a union of (possibly translated) subtori of $\text{Hom}(G, \mathbb{C}^*)$.

1.5. Classification up to Malcev completion. Our first main result gives the classification of quasi-Kähler, 1-formal, 3-manifold groups, at the level of Malcev completions.

Theorem 1.1. *Let G be the fundamental group of a compact, connected, orientable 3-manifold. Assume G is 1-formal. Then the following are equivalent.*

- (1) *The Malcev completion of G is isomorphic to the Malcev completion of a quasi-Kähler group.*
- (2) *The Malcev completion of G is isomorphic to the Malcev completion of the fundamental group of S^3 , $\#^n S^1 \times S^2$, or $S^1 \times \Sigma_g$, with $g \geq 1$.*

Thus, when viewed through the prism of their Malcev Lie algebras, all quasi-Kähler, 1-formal, 3-manifold groups look like either a trivial group, a free group, or an infinite cyclic group times a surface group.

The proof relies heavily on the structure of the resonance varieties of quasi-Kähler, 1-formal groups, as described in [13], and the resonance varieties of 3-manifold groups, as described in [14]. An important role is played by the 0-isotropic and 1-isotropic subspaces of $H^1(G, \mathbb{C})$, and their position with respect to the resonance variety $R_1^1(G)$.

1.6. Corank and isotropy index. For a connected CW-complex X with finite 1-skeleton, the *isotropy index* $i(X)$ is the maximum dimension of those subspaces $W \subseteq H^1(X, \mathbb{C})$ for which the restriction of \cup_X to $W \wedge W$ is zero. When X is a smooth complex algebraic variety, these subspaces W are precisely the maximal isotropic subspaces considered by Catanese [5] and Bauer [2].

For a finitely generated group G , put $i(G) = i(K(G, 1))$; it is readily seen that $i(X) = i(\pi_1(X))$. It turns out that the isotropy index $i(G)$ depends only on $\mathfrak{m}(G)$, the Malcev completion of the group.

The *corank* of G is the largest integer n such that there is an epimorphism $G \twoheadrightarrow F_n$. This is a subtler invariant, depending on more information, even for 1-formal groups. For example, if $G = \pi_1(\#^3 \mathbb{R}P^2)$, then G is 1-formal, $\mathfrak{m}(G) = \mathfrak{m}(F_2)$, yet $\text{corank}(G) < \text{corank}(F_2)$. Restricting to the class of (1-formal) quasi-Kähler groups, we show that the corank and the isotropy index coincide; thus, such a phenomenon cannot happen within this class. The result below improves Lemma 6.16 from [13].

Theorem 1.2. *Let G be a quasi-Kähler, 1-formal group. Then $\text{corank}(G) = i(G)$. In particular, the corank of G depends only on its Malcev completion.*

The 1-formality assumption is crucial here. For example, let G be the fundamental group of the Heisenberg nilmanifold. As noted in §1.2 (see also Morgan [27, p. 203]), G is a quasi-Kähler group. Nevertheless, $\text{corank}(G) = 1$, yet $i(G) = 2$.

Suppose now our quasi-Kähler, 1-formal group is also a 3-manifold group. Using Theorems 1.1 and 1.2, we can compute its corank explicitly.

Corollary 1.3. *Let G be a quasi-Kähler, 1-formal, 3-manifold group. Then the Malcev completion of G is that of either 1 , F_n , or $\mathbb{Z} \times \pi_1(\Sigma_g)$, in which case $\text{corank}(G)$ equals 0 , n , or g . In particular, if $b_1(G) = 2k$, then $\mathfrak{m}(G) = \mathfrak{m}(F_{2k})$, and $\text{corank}(G) = b_1(G)$.*

1.7. Quasi-homogeneous surface singularities. Up to this point, we have only discussed quasi-Kähler, 3-manifold groups which are 1-formal. But there are many groups of this sort which are not 1-formal. A rich supply of such groups occurs in the context of isolated surface singularities with \mathbb{C}^* -action.

Let $(X, 0)$ be such a singularity, and let M be the associated singularity link (a closed, oriented 3-manifold). Then clearly $G = \pi_1(M)$ is a quasi-projective, 3-manifold group. Yet we show in Proposition 7.1 that G is not 1-formal, provided $b_1(M) > 0$.

In the same vein, we construct in Proposition 7.2 irreducible, smooth *affine* surfaces U , with non-1-formal fundamental group G . To the best of our knowledge, these are the first examples of this kind. Note that the Deligne weight filtration subspace $W_1 H^1(U, \mathbb{C})$ must be non-trivial, since otherwise G would be 1-formal, according to Morgan's results from [27].

Computing the characteristic varieties of groups which are not 1-formal can be a very arduous task. Nevertheless, this can be done in our context: in Proposition 8.1, we give a precise description of all positive-dimensional irreducible components of $V_1^1(M)$, in the case when M is the link of an isolated surface singularity with \mathbb{C}^* -action. In particular, we describe the analytic germs at the origin of those components, solely in terms of $b_1(M)$. Detailed computations are carried out in the case when $M = \Sigma(a_1, \dots, a_n)$ is a Brieskorn manifold.

1.8. Organization of the paper. In Section 2 we review some basic material on Malcev completions and 1-formality, while in Section 3 we review cohomology jumping loci and the tangent cone theorem for 1-formal groups. In Section 4, we discuss the characteristic and resonance varieties of quasi-Kähler manifolds and closed 3-manifolds, with particular attention to their isotropicity properties.

Section 5 is devoted to a proof of Theorem 1.1, based on the different qualitative properties of resonance varieties in the quasi-Kähler and 3-dimensional settings. We start Section 6 with a discussion of the corank and isotropy index; the rest of the section is devoted to proving Theorem 1.2 and Corollary 1.3, and to an application.

In Section 7, we study the formality properties of links of quasi-homogeneous surface singularities. And finally, in Section 8, we discuss the translated components in the characteristic varieties of these singularity links.

2. MALCEV COMPLETION AND 1-FORMALITY

2.1. Malcev completion. Let G be a group. The lower central series of G is defined inductively by $\gamma_1 G = G$ and $\gamma_{k+1} G = [\gamma_k G, G]$, where $[x, y] = xyx^{-1}y^{-1}$. The *associated graded Lie algebra*, $\text{gr}(G)$, is the direct sum of the successive LCS quotients, $\text{gr}(G) = \bigoplus_{k \geq 1} \gamma_k G / \gamma_{k+1} G$, with Lie bracket induced from the group commutator; see [24].

In [35, Appendix A], Quillen associates to any group G a pronilpotent, rational Lie algebra, $\mathfrak{m}(G)$, called the *Malcev completion* of G . The Malcev Lie algebra comes endowed with a decreasing, complete, \mathbb{Q} -vector space filtration, such that the associated graded Lie algebra, $\text{gr}(\mathfrak{m}(G))$, is isomorphic to $\text{gr}(G) \otimes \mathbb{Q}$. For precise definitions and basic properties of Malcev Lie algebras, we refer to [32].

Now suppose G is a finitely generated group. We then have the following result of Sullivan [40].

Lemma 2.1. *The Malcev completion $\mathfrak{m}(G)$ determines the corestriction to its image of the cup-product map $\cup_G: H^1(G, \mathbb{Q}) \wedge H^1(G, \mathbb{Q}) \rightarrow H^2(G, \mathbb{Q})$.*

In particular, the first Betti number, $b_1(G) = \dim H^1(G, \mathbb{Q})$, is determined by $\mathfrak{m}(G)$. Note also that if $G = \pi_1(X)$, then \cup_G and \cup_X have the same corestriction.

We will also need the following fact, extracted from [40].

Lemma 2.2. *Let $\phi: G' \rightarrow G$ be a group homomorphism. If $\phi^1: H^1(G, \mathbb{Q}) \rightarrow H^1(G', \mathbb{Q})$ is an isomorphism, and $\phi^2: H^2(G, \mathbb{Q}) \rightarrow H^2(G', \mathbb{Q})$ is a monomorphism, then ϕ induces an isomorphism $\phi_*: \mathfrak{m}(G') \rightarrow \mathfrak{m}(G)$ between Malcev completions.*

Suppose now $G = \pi_1(X)$ and $G' = \pi_1(X')$. Let $f: X' \rightarrow X$ be a continuous map, and set $\phi = f_\#: G' \rightarrow G$. Recall that a classifying map, $\kappa: X \rightarrow K(G, 1)$, induces an isomorphism on H^1 and a monomorphism on H^2 . Hence, ϕ satisfies the cohomological hypotheses of Lemma 2.2 whenever $f^*: H^*(X, \mathbb{Q}) \rightarrow H^*(X', \mathbb{Q})$ satisfies them.

2.2. 1-formality. As before, let X be a connected CW-complex, with finite 1-skeleton. Following K.-T. Chen [6], define the (*rational*) *holonomy Lie algebra* of X as

$$(1) \quad \mathfrak{h}(X) = \text{Lie}(H_1(X, \mathbb{Q})) / \text{ideal}(\text{im}(\partial_X)),$$

where $\text{Lie}(H_1(X, \mathbb{Q}))$ is the free (graded) Lie algebra over \mathbb{Q} , generated in degree 1 by $H_1(X; \mathbb{Q})$, and $\partial_X: H_2(X, \mathbb{Q}) \rightarrow H_1(X, \mathbb{Q}) \wedge H_1(X, \mathbb{Q})$ is the dual of the cup-product map, \cup_X . Since its defining ideal is homogeneous (in fact, quadratic), $\mathfrak{h}(X)$ inherits a natural grading from the free Lie algebra. Note that $\mathfrak{h}(X)$ depends only on the corestriction of \cup_X to its image. It follows that $\mathfrak{h}(X) = \mathfrak{h}(\pi_1(X))$, where $\mathfrak{h}(G) := \mathfrak{h}(K(G, 1))$.

Now let G be a finitely generated group, and let $\widehat{\mathfrak{h}}(G)$ be the completion of $\mathfrak{h}(G)$ with respect to the degree filtration. Following Sullivan [40], we say G is *1-formal* if $\mathfrak{m}(G) \cong \widehat{\mathfrak{h}}(G)$, as (complete) filtered Lie algebras. Equivalently, $\mathfrak{m}(G)$ is filtered Lie isomorphic to the degree completion of a quadratic Lie algebra.

It follows from the definitions that the Malcev completion of a 1-formal group G is determined by the corestriction of \cup_G to its image; or, the corestriction of \cup_X to its image, if $G = \pi_1(X)$, where X is a CW-complex with finite 1-skeleton.

Example 2.3. Let F_n be the free group of rank n . Then, as shown by Witt and Magnus, $\text{gr}(F_n) \otimes \mathbb{Q} = \mathbb{L}_n$, the free Lie algebra of rank n . It is readily checked that $\mathfrak{h}(F_n) = \mathbb{L}_n$, and $\mathfrak{m}(F_n) = \widehat{\mathbb{L}}_n$. In particular, F_n is 1-formal.

Example 2.4. Let Σ_g be the Riemann surface of genus $g \geq 1$, with fundamental group $\Pi_g = \pi_1(\Sigma_g)$ generated by $x_1, y_1, \dots, x_g, y_g$, subject to the relation $[x_1, y_1] \cdots [x_g, y_g] = 1$. It follows from [8] that the group Π_g is 1-formal. Using (1), it is readily checked that $\mathfrak{h}(\Pi_g)$ is the quotient of the free Lie algebra on $x_1, y_1, \dots, x_g, y_g$ by the ideal generated by $[x_1, y_1] + \cdots + [x_g, y_g]$.

3. COHOMOLOGY JUMPING LOCI

3.1. Characteristic varieties. Let X be a connected CW-complex with finitely many cells in each dimension. Let $G = \pi_1(X)$ be the fundamental group of X , and $\mathbb{T}(G) = \text{Hom}(G, \mathbb{C}^*)$ its character variety. Every character $\rho \in \mathbb{T}(G)$ determines a rank 1 local system, \mathbb{C}_ρ , on X . The *characteristic varieties* of X are the jumping loci for cohomology with coefficients in such local systems:

$$(2) \quad V_d^i(X) = \{\rho \in \mathbb{T}(G) \mid \dim H^i(X, \mathbb{C}_\rho) \geq d\}.$$

The varieties $V_d(X) = V_d^1(X)$ depend only on $G = \pi_1(X)$, so we sometimes denote them as $V_d(G)$. Here is an alternate description, along the lines of [12].

Let 1 be the identity of the algebraic group $\mathbb{T}(G)$, and let $\mathbb{T}^0(G)$ be the connected component containing 1. Set $n = b_1(G)$, and let $G_{\text{abf}} = \mathbb{Z}^n$ be the maximal torsion-free abelian quotient of G . The group algebra $\mathbb{C}G_{\text{abf}}$ may be identified with $\Lambda_n = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, the coordinate ring of the algebraic torus $\mathbb{T}^0(G) = (\mathbb{C}^*)^n$. Finally, set $A_G = \mathbb{C}G_{\text{abf}} \otimes_{\mathbb{C}G} I_G$, where $\mathbb{C}G$ is the group algebra of G , with augmentation ideal I_G . Then, for all $d < n$ (and away from 1 if $d \geq n$),

$$(3) \quad V_d(G) \cap \mathbb{T}^0(G) = V(E_d(A_G)),$$

the subvariety of $\mathbb{T}^0(G)$ defined by $E_d(A_G)$, the ideal of codimension d minors of a presentation matrix for A_G ; see [12, Proposition 2.4]. Such a matrix can be computed explicitly from a finite presentation for G , by means of the Fox calculus.

In favorable situations, the intersection of $V_1(G)$ with the torus $\mathbb{T}^0(G)$ is a hypersurface, defined by the vanishing of the *Alexander polynomial*, Δ^G . More precisely, let $\Delta^G \in \Lambda_n$ be the greatest common divisor of all elements of $E_1(A_G)$, up to units in Λ_n . Suppose the Alexander ideal $E_1(A_G)$ is *almost principal*, i.e., there is an integer $k > 0$ such that $I_G^k \cdot (\Delta^G) \subset E_1(A_G)$. We then have $V_1(G) \cap \mathbb{T}^0(G) = V(\Delta^G)$, at least away from 1. For more details, see [12, §6.1].

3.2. Resonance varieties. Consider now the cohomology algebra $H^*(X, \mathbb{C})$. Left-multiplication by an element $x \in H = H^1(X, \mathbb{C})$ yields a cochain complex $(H^*(X, \mathbb{C}), \lambda_x)$. The *resonance varieties* of X are the jumping loci for the homology of this complex:

$$(4) \quad R_d^i(X) = \{x \in H \mid \dim H^i(H^*(X, \mathbb{C}), \lambda_x) \geq d\}.$$

The homogeneous varieties $R_d(X) = R_d^1(X)$ depend only on $G = \pi_1(X)$, so we sometimes denote them by $R_d(G)$. In view of Lemma 2.1, the varieties $R_d(G)$ depend only on the Malcev completion, $\mathfrak{m}(G)$.

3.3. The tangent cone theorem. Identify $H^1(X, \mathbb{C})$ with $T_1\mathbb{T}(G) = \text{Hom}(G, \mathbb{C})$, the Lie algebra of the algebraic group $\mathbb{T}(G)$, and let $\exp: T_1\mathbb{T}(G) \rightarrow \mathbb{T}(G)$ be the exponential map. Under this identification, the tangent cone to $V_d^i(X)$ at the origin, $TC_1(V_d^i(X))$, is contained in $R_d^i(X)$, see Libgober [21]. In general, though, this inclusion is strict.

The main link between the 1-formality property of a group and its cohomology jumping loci is provided by the following theorem.

Theorem 3.1 (Dimca–Papadima–Suciu [13], Theorem A). *Suppose G is a 1-formal group. Then, for each $d \geq 1$, the exponential map induces a complex analytic isomorphism between the germ at 0 of $R_d(G)$ and the germ at 1 of $V_d(G)$. Consequently, $TC_1(V_d(G)) = R_d(G)$.*

4. ISOTROPICITY PROPERTIES

4.1. Isotropic subspaces. Let G be a finitely generated group. We say that a non-zero subspace $W \subseteq H^1(G, \mathbb{C})$ is p -isotropic ($p = 0$ or 1) with respect to the cup-product map $\cup_G: \bigwedge^2 H^1(G, \mathbb{C}) \rightarrow H^2(G, \mathbb{C})$ if the restriction of \cup_G to $\bigwedge^2 W$ has rank p .

The motivation for this definition comes from the case when $G = \pi_1(C)$ is the fundamental group of a non-simply-connected smooth complex curve C . Set $W = H^1(G, \mathbb{C})$. If C is non-compact, then $G = F_n$, for some $n > 0$, and so W is 0-isotropic. On the other hand, if C is compact, then $G = \Pi_g$, for some $g > 0$, and so W is 1-isotropic.

Note that the p -isotropy property depends only on the Malcev completion, by Lemma 2.1.

4.2. Characteristic varieties of quasi-Kähler manifolds. Let X be a quasi-Kähler manifold, with fundamental group $G = \pi_1(X)$. As mentioned in §1.4, it is known that $V_1(G)$ is a union of (possibly translated) subtori of $\mathbb{T}(G) = \text{Hom}(G, \mathbb{C}^*)$. Let us describe in more detail those components passing through the identity $1 \in \mathbb{T}(G)$.

Following Arapura [1, p. 590], we say that a map $f: X \rightarrow C$ to a connected, smooth complex curve C is *admissible* if f is holomorphic and surjective, and has a holomorphic, surjective extension with connected fibers to smooth compactifications, $\bar{f}: \bar{X} \rightarrow \bar{C}$, obtained by adding divisors with normal crossings. In particular, the generic fiber of f is connected, and the induced homomorphism, $f_{\#}: \pi_1(X) \rightarrow \pi_1(C)$, is onto. Two such maps, $f: X \rightarrow C$ and $f': X \rightarrow C'$, are said to be equivalent if there is an isomorphism $\psi: C \rightarrow C'$ such that $f' = \psi \circ f$. Proposition V.1.7 from [1] can now be stated, as follows.

Theorem 4.1 (Arapura [1]). *There is a bijection between the set of positive-dimensional irreducible components of $V_1(G)$ containing 1, and the set of equivalence classes of admissible maps $f: X \rightarrow C$ with $\chi(C) < 0$. This bijection associates to f the subtorus $S_f = f^*\mathbb{T}(\pi_1(C)) \subseteq \mathbb{T}^0(G)$.*

4.3. Resonance varieties of quasi-Kähler manifolds. As above, let X be a quasi-Kähler manifold. Let us assume now that $G = \pi_1(X)$ is 1-formal. Using the aforementioned results of Arapura, in conjunction with Theorem 3.1, we establish in [13, Theorem C] a bijection between the positive-dimensional irreducible components of $R_1(G)$ and admissible maps $f: X \rightarrow C$ with $\chi(C) < 0$. Under this bijection, an admissible map f corresponds to the linear subspace $W_f = f^*H^1(C, \mathbb{C}) = T_1(S_f) \subseteq H^1(G, \mathbb{C})$. As explained in detail in [11], this correspondence is closely related to the results of Catanese [5] and Bauer [2] on isotropic subspaces.

With this notation, Proposition 7.2(3) from [13] can be reformulated in a more precise fashion, as follows.

Theorem 4.2 (Dimca–Papadima–Suciu [13]). *Let X be a quasi-Kähler manifold, with 1-formal fundamental group G . Suppose $f: X \rightarrow C$ is an admissible map onto a smooth curve C with $\chi(C) \leq 0$. There are then two possibilities.*

- (1) W_f is a 0-isotropic subspace, which happens precisely when C is non-compact.
- (2) W_f is a 1-isotropic subspace, which happens precisely when C is compact.

4.4. Jumping loci of 3-manifolds. Let M be a compact, connected, orientable 3-manifold, with fundamental group π , and Alexander polynomial Δ^π . As shown by McMullen in [25, Theorem 5.1], $I_\pi^2 \cdot (\Delta^\pi) \subset E_1(A_\pi)$, and so the Alexander ideal $E_1(A_\pi)$ is almost principal. From the discussion at the end of §3.1, we have

$$(5) \quad V_1(M) \cap \mathbb{T}^0(\pi) = V(\Delta^\pi),$$

at least away from 1.

Now fix an orientation on M ; that is to say, pick a generator $[M] \in H_3(M, \mathbb{Z}) \cong \mathbb{Z}$. With this choice, the cup product on M and Kronecker pairing determine an alternating 3-form μ_M on $H^1(M, \mathbb{Z})$, given by $\mu_M(x, y, z) = \langle x \cup y \cup z, [M] \rangle$.

In coordinates, we can write this form as follows. Choose a basis $\{e_1, \dots, e_n\}$ for $H_1(M, \mathbb{C})$. Let $\{e_1^*, \dots, e_n^*\}$ be the Kronecker dual basis for $H^1(M, \mathbb{C})$, and let $\{e_1^\vee, \dots, e_n^\vee\}$ be the Poincaré dual basis for $H^2(M, \mathbb{C})$, defined by $\langle e_i^\vee \cup e_j^*, [M] \rangle = \delta_{ij}$. Set $\mu_{ijk} = \langle e_i^* \cup e_j^* \cup e_k^*, [M] \rangle$. The 3-form $\eta = \mu_M \otimes \mathbb{C}$ is then given by

$$(6) \quad \eta = \sum_{i,j,k} \mu_{ijk} e_i \wedge e_j \wedge e_k.$$

Remark 4.3. In [39], Sullivan showed that *any* alternating 3-form $\eta \in \bigwedge^3(\mathbb{Q}^n)^*$ can be realized as $\eta = \mu_M \otimes \mathbb{Q}$, for some closed, connected, oriented 3-manifold M with $b_1(M) = n$. He also showed that, if X is a complex algebraic surface with an isolated singularity, and M is the link of that singularity, then $\mu_M = 0$.

We will need the following result from [14], which provides information on the first resonance variety of a 3-manifold, and its isotropicity properties.

Theorem 4.4 (Dimca–Suciu [14]). *Let M be a closed, orientable 3-manifold. Then:*

- (1) $H^1(M, \mathbb{C})$ is not 1-isotropic.
- (2) If $b_1(M)$ is even, then $R_1(M) = H^1(M, \mathbb{C})$.
- (3) If $R_1(M)$ contains a 0-isotropic hyperplane, then $R_1(M) = H^1(M, \mathbb{C})$.

In the next remark, we provide an alternative proof of Part (2) of this theorem, and further information on the remaining case, when $b_1(M)$ is odd.

Remark 4.5. Assume $n = b_1(M) > 0$, and let $x \in H^1(M, \mathbb{C})$ be a non-zero element. By definition, $x \notin R_1(M)$ if and only if the linear map $\lambda_x: H^1(M, \mathbb{C}) \rightarrow H^2(M, \mathbb{C})$ has rank $n - 1$. Choose a basis $\{e_1, \dots, e_n\}$ for $H_1(M, \mathbb{C})$ so that $x = e_n^*$. With notation as above, identify the subspaces spanned by $\{e_1^*, \dots, e_{n-1}^*\}$ and $\{e_1^\vee, \dots, e_{n-1}^\vee\}$ with \mathbb{C}^{n-1} , and let $\lambda: \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}$ be the restriction of λ_x to those subspaces. The matrix of λ is skew-symmetric, with entries μ_{ijn} , for $1 \leq i, j < n$; moreover, λ and λ_x have the same rank. Hence, $x \notin R_1(M)$ if and only if $\det(\lambda) \neq 0$. There are two cases to consider.

If $n = b_1(M)$ is even, it follows that $\det(\lambda) = 0$. Therefore, $R_1(M) = H^1(M, \mathbb{C})$.

If $n = b_1(M)$ is odd, say $n = 2m + 1$, it follows that $R_1(M) \neq H^1(M, \mathbb{C})$ if and only if

$$(7) \quad \eta = (e_1 \wedge e_2 + \cdots + e_{2m-1} \wedge e_{2m}) \wedge e_{2m+1} + \sum_{1 \leq i, j, k \leq 2m} \mu_{ijk} e_i \wedge e_j \wedge e_k,$$

with respect to some basis $\{e_1, \dots, e_n\}$. This means precisely that the Poincaré duality algebra $H^*(M, \mathbb{C})$ is generic, in the sense of definition (3') from [4, p. 463].

5. QUASI-KÄHLER 3-MANIFOLD GROUPS

This section is devoted to the proof of Theorem 1.1 from the Introduction, namely, the nontrivial implication, (1) \Rightarrow (2). Recall Σ_g is the Riemann surface of genus $g > 0$, with fundamental group Π_g . It is enough to prove the following lemma.

Lemma 5.1. *Let M be a closed, orientable 3-manifold, and let X be a quasi-Kähler manifold. Assume both $\pi = \pi_1(M)$ and $G = \pi_1(X)$ are 1-formal, and their Malcev completions are isomorphic. Then $\mathfrak{m}(\pi)$ is isomorphic to either 0, $\mathfrak{m}(F_n)$, or $\mathfrak{m}(\mathbb{Z} \times \Pi_g)$.*

Proof. The proof is broken into four steps.

Step 1. Recall from §2.1 that the Malcev completion of a group determines its first Betti number. Since, by assumption, $\mathfrak{m}(\pi) \cong \mathfrak{m}(G)$, we must have $b_1(\pi) = b_1(G)$. Let n denote this common Betti number. We first dispose of some easy cases.

Suppose $\mu_M = 0$. Then $\cup_M = \cup_{M_n}$, where $M_0 = S^3$ and $M_n = \#^n S^1 \times S^2$ for $n > 0$. Since $\pi_1(M_n) = F_n$ is 1-formal, $\mathfrak{m}(\pi) = \mathfrak{m}(F_n)$, and we are done. So we may assume for the rest of the proof that $\mu_M \neq 0$. This immediately forces $n \geq 2$.

Now suppose $n = 3$. Since $\mu_M \neq 0$, we must have $\mu_M \otimes \mathbb{Q} = \mu_{T^3} \otimes \mathbb{Q}$, where $T^3 = S^1 \times \Sigma_1$ is the 3-torus. Since $\pi_1(T^3) = \mathbb{Z}^3$ is 1-formal, $\mathfrak{m}(\pi) = \mathfrak{m}(\mathbb{Z}^3)$, and we are done. Thus, for the rest of the proof, we may assume $n \geq 4$.

Step 2. This step takes care of the case when $R_1(X) = H^1(X, \mathbb{C})$. From the discussion in §4.3, we know there is an admissible map $f: X \rightarrow C$ such that $W_f = H^1(X, \mathbb{C})$. According to Theorem 4.2, the subspace W_f is either 0-isotropic or 1-isotropic.

On the other hand, since $\mathfrak{m}(\pi) = \mathfrak{m}(G)$, we have $H^1(X, \mathbb{C}) = H^1(M, \mathbb{C})$. In view of Theorem 4.4(1), the space $H^1(M, \mathbb{C})$ must be 0-isotropic. Therefore, W_f is 0-isotropic. Hence, by Theorem 4.2, the curve C is non-compact.

By construction, the map $f^1: H^1(C, \mathbb{Q}) \rightarrow H^1(X, \mathbb{Q})$ is an isomorphism. Since C is non-compact, $f^2: H^2(C, \mathbb{Q}) \rightarrow H^2(X, \mathbb{Q})$ is trivially a monomorphism. Applying now Lemma 2.2, we conclude that G has the same Malcev completion as $\pi_1(C) = F_n$.

Step 3. We now may assume that $R_1(X) \neq H^1(X, \mathbb{C})$, which implies $R_1(M) \neq H^1(M, \mathbb{C})$. From Theorem 4.4(2), we infer that $n = 2g + 1$, for some $g \geq 2$. We claim that, in this case, $R_1(X)$ is a hyperplane, defined over \mathbb{Q} .

Indeed, Theorem 3.1 guarantees that $R_1(X) = TC_1(V_1(X))$, and likewise for M . On the other hand, we know from (5) that $V_1(M) \cap \mathbb{T}^0(\pi) = V(\Delta^\pi)$, away from 1. Since $n \geq 4$, both subvarieties of $\mathbb{T}^0(\pi)$ contain 1; see [23, Proposition 3.11]. In particular, $\Delta^\pi \neq 0, 1$. Hence, all irreducible components of $R_1(M)$ have codimension 1.

The same property holds for $R_1(X)$. By the discussion preceding Theorem 4.2, all irreducible components of $R_1(X)$ must be hyperplanes, defined over \mathbb{Q} . Using Theorem 4.2 from [12] (which continues to hold in the quasi-Kähler case), we infer that any two distinct components of $R_1(X)$ intersect only at 0. Since $n \geq 3$, the variety $R_1(X)$ must be irreducible, and this proves our claim.

Step 4. By Theorems 4.2 and 4.4(3), $R_1(X) = f^*H^1(C, \mathbb{Q})$, the curve C is compact of genus g , and $\omega := f^*(\omega_0) \neq 0$, where $\omega_0 \in H^2(C, \mathbb{Q})$ is the generator corresponding to the complex orientation.

Pick a map $h: X \rightarrow S^1$ with the property that $t := h^*(\eta_0) \notin R_1(X)$, where $\eta_0 \in H^1(S^1, \mathbb{Q})$ is the orientation generator. Set $F = (h, f): X \rightarrow S^1 \times C$. By construction, $F^1: H^1(S^1 \times C, \mathbb{Q}) \rightarrow H^1(X, \mathbb{Q})$ is an isomorphism. We claim that $F^2: H^2(S^1 \times C, \mathbb{Q}) \rightarrow H^2(X, \mathbb{Q})$ is a monomorphism.

To verify the claim, decompose the source according to the Künneth formula,

$$H^2(S^1 \times C, \mathbb{Q}) = \mathbb{Q} \cdot \omega_0 \oplus (\mathbb{Q} \cdot \eta_0 \otimes H^1(C, \mathbb{Q})),$$

and consider the restriction of F^2 to each of the two summands.

Firstly, F^2 is injective on the second summand. For otherwise, we could find a non-zero element $z \in R_1(X)$ with the property that $t \cup z = 0$. But this contradicts our assumption that $t \notin R_1(X)$.

Secondly, we must check that $\omega \notin t \cup R_1(X)$. Supposing the contrary, let $\omega = t \cup x$, with $0 \neq x \in R_1(X)$. By Poincaré duality on C , we may find $y \in R_1(X)$ such that $\omega = y \cup x$. Thus, $(t - y) \cup x = 0$, and so $t - y \in R_1(X)$, contradicting again the choice of t .

The claim is now proved. Using Lemma 2.2, we conclude that $\mathfrak{m}(G) = \mathfrak{m}(\mathbb{Z} \times \Pi_g)$, and we are done. \square

6. CORANK AND ISOTROPY INDEX

6.1. Two numerical invariants. To every finitely generated group G , we associated in §1.6 two non-negative integers—the corank and the isotropy index:

$$(8) \quad \begin{aligned} \text{corank}(G) &= \max\{n \mid \exists \text{ epimorphism } G \twoheadrightarrow F_n\}, \\ i(G) &= \max\{d \mid \exists W \subseteq H^1(G, \mathbb{C}), \dim W = d, \cup_G|_{W \wedge W} = 0\}. \end{aligned}$$

Clearly, both these invariants are bounded above by the first Betti number $b_1(G)$. If there is an epimorphism $G \twoheadrightarrow H$, then $\text{corank}(G) \geq \text{corank}(H)$. In particular, if $b_1(G) > 0$, then $\text{corank}(G) > 0$. Moreover, if $\phi: G \twoheadrightarrow F_n$ is an epimorphism, the restriction of \cup_G to $\phi^1(H^1(F_n, \mathbb{C}))$ vanishes; thus,

$$(9) \quad \text{corank}(G) \leq i(G).$$

The corank of a 3-manifold group has been studied by Harvey [17], Leininger and Reid [20], and Sikora [37]. In particular, if M is a closed 3-manifold, the corank of $\pi_1(M)$ equals the “cut number” of M .

In view of Lemma 2.1, the isotropy index $i(G)$ depends only on the Malcev completion $\mathfrak{m}(G)$. In general, though, the corank of G is not determined by $\mathfrak{m}(G)$, even when G is 1-formal. Adapting Example 6.18 from [13] to our situation illustrates this phenomenon.

Example 6.1. Let $N = \#^3\mathbb{R}\mathbb{P}^2$ be the non-orientable surface of genus 3. It is readily seen that N has the rational homotopy type of $S^1 \vee S^1$. Hence, N is a formal space, and $G = \pi_1(N)$ is a 1-formal group, with $b_1(G) = 2$. Moreover, $\mathfrak{m}(G) \cong \mathfrak{m}(F_2)$.

Suppose there is an epimorphism $\phi: G \twoheadrightarrow F_2$. Then $\phi^*H^1(F_2, \mathbb{Z}_2)$ is a 2-dimensional subspace of $H^1(G, \mathbb{Z}_2) = \mathbb{Z}_2^3$, isotropic with respect to \cup_G . But this is impossible, by Poincaré duality with \mathbb{Z}_2 coefficients in $N = K(G, 1)$. Hence, $\text{corank}(G) = 1$, though of course, $\text{corank}(F_2) = 2$.

6.2. Proof of Theorem 1.2. Taking into account inequality (9), it is enough to prove the following lemma.

Lemma 6.2. *Let X be a quasi-Kähler manifold. Assume $G = \pi_1(X)$ is 1-formal. Then $\text{corank}(G) \geq i(G)$.*

Proof. Set $d := i(G)$. We may assume $d \geq 2$. Indeed, if $d = 0$ we are done, and if $d = 1$ then $b_1(G) \geq 1$, which implies $\text{corank}(G) \geq 1$.

Let $W \subseteq H^1(X, \mathbb{C})$ be a d -dimensional, 0-isotropic subspace. Clearly, $W \subseteq R_1(X)$, since $d \geq 2$. By the discussion preceding Theorem 4.2, there is an admissible map, $f: X \rightarrow C$, onto a smooth complex curve with $\chi(C) < 0$, such that $W \subseteq W_f$. In particular, we have an epimorphism $f_\sharp: G \twoheadrightarrow \pi_1(C)$. The argument splits according to the two cases from Theorem 4.2. Set $b = b_1(C)$.

- (1) *C is non-compact.* In this case, $\pi_1(C) = F_b$. Thus, we have a surjection $f_\sharp: G \twoheadrightarrow F_b$, and so $\text{corank}(G) \geq b$. On the other hand, $b = \dim W_f \geq d$.
- (2) *C is compact.* In this case, $b = 2g$, where g is the genus of C , and W is a 0-isotropic subspace of the 1-isotropic space $H^1(C, \mathbb{C})$. Hence, $d \leq g$. Composing the obvious epimorphism $\pi_1(C) \twoheadrightarrow F_g$ with f_\sharp , we obtain a surjection $G \twoheadrightarrow F_g$. Hence, $g \leq \text{corank}(G)$.

In either case, we conclude that $d \leq \text{corank}(G)$, and this ends the proof. \square

6.3. Proof of Corollary 1.3. We need to determine the corank of a quasi-Kähler, 1-formal, 3-manifold group. In view of Theorems 1.1 and 1.2, it is enough to compute $i(S^3)$, $i(\#^n S^1 \times S^2)$, and $i(S^1 \times \Sigma_g)$. We do this next.

Lemma 6.3. *The following hold.*

- (1) *If $M = S^3$ or $M = \#^n S^1 \times S^2$, then $i(M) = b_1(M)$.*
- (2) *If $M = S^1 \times \Sigma_g$, with $g \geq 1$, then $i(M) = \frac{1}{2}(b_1(M) - 1)$.*

Proof. If $M = S^3$ or $M = \#^n S^1 \times S^2$, then $\cup_M = 0$, and so $i(M) = b_1(M)$. If $M = S^1 \times \Sigma_1$, it is readily seen that $i(M) = 1$, as asserted.

In the remaining case, $M = S^1 \times \Sigma_g$, with $g \geq 2$, we have to check that $g = i(M)$. By considering a maximal 0-isotropic subspace $W \subseteq H^1(\Sigma_g, \mathbb{C})$, we see that $g \leq i(M)$. A standard argument now shows that every 0-isotropic subspace $W \subseteq H^1(M, \mathbb{C})$ of dimension $i(M)$ is contained in $H^1(\Sigma_g, \mathbb{C})$; thus, $g \geq i(M)$. \square

6.4. On a certain class of 3-manifolds. We conclude this section with an application of Corollary 1.3 to low-dimensional topology. The next result identifies the class of 3-manifolds whose fundamental groups satisfy certain properties.

Corollary 6.4. *Let M be a closed, orientable 3-manifold, with fundamental group $G = \pi_1(M)$. The following are equivalent.*

- (1) *The group G is quasi-Kähler, 1-formal, presentable by commutator-relators, and has even first Betti number.*
- (2) *The manifold M is either S^3 or $\#^{2k} S^1 \times S^2$.*

Proof. The implication (2) \Rightarrow (1) is clear. So assume G satisfies the conditions from (1). By Corollary 1.3, $\text{corank}(G) = b_1(G) = n$, with n even. Thus, we have an epimorphism $G \twoheadrightarrow F_n$. Since G is presented by commutator-relators, there is another epimorphism, $F_n \twoheadrightarrow G$. As is well known, finitely generated free groups are Hopfian, i.e., any epimorphism $F_n \twoheadrightarrow F_n$ must be an isomorphism, see [24, Theorem 2.13]. Hence, $G = F_n$. Assertion (2) now follows from standard 3-manifold theory [18] and the Poincaré conjecture, proved by Perelman in [33, 34]. \square

7. QUASI-HOMOGENEOUS SURFACE SINGULARITIES AND 1-FORMALITY

Let $(X, 0)$ be a complex, normal, isolated surface singularity. We may embed $(X, 0)$ in $(\mathbb{C}^n, 0)$, for some sufficiently large n . Then, for sufficiently small $\epsilon > 0$, the singularity link, $M = X \cap S_\epsilon^{2n-1}$, is a closed, oriented, smooth 3-manifold, whose oriented diffeomorphism type is independent of the choices made.

Assume now $(X, 0)$ is a quasi-homogeneous singularity. We may then represent $(X, 0)$ by an affine surface X with a “good” \mathbb{C}^* -action. By definition, this means the isotropy groups \mathbb{C}_x^* are finite, for all $x \neq 0$, and the induced action of \mathbb{C}^* on the finite-dimensional vector space $\mathfrak{m}_{(X,0)}/\mathfrak{m}_{(X,0)}^2$ has only positive weights. For details on this subject, we refer to Orlik and Wagreich [31], Looijenga [22, p. 175], and also [9, p. 52, 66].

Proposition 7.1. *Let $(X, 0)$ be a quasi-homogeneous isolated surface singularity, and let M be the corresponding singularity link. Then, the fundamental group $G = \pi_1(M)$ is a quasi-projective, 3-manifold group. Moreover,*

- (1) *If $b_1(M) = 0$, then M is formal, and so G is 1-formal.*
- (2) *If $b_1(M) > 0$, then G is not 1-formal.*

Proof. Set $X^* = X \setminus \{0\}$. Clearly, X^* is a smooth, quasi-projective variety. Moreover, the singularity link M is homotopy equivalent to X^* . The first assertion follows at once.

Consider now the orbit space, $C = X^*/\mathbb{C}^*$. This is a smooth projective curve of genus $g = \frac{1}{2}b_1(M)$. In particular, $b_1(M)$ must be even.

If $b_1(M) = 0$, then M is a rational homology 3-sphere, i.e., $H^*(M, \mathbb{Q}) \cong H^*(S^3, \mathbb{Q})$. A result of Halperin and Stasheff [16, Corollary 5.16] implies that M is formal.

Now assume $b_1(M) > 0$. Then the curve $C = X^*/\mathbb{C}^*$ has genus $g \geq 1$. Let $p: X^* \rightarrow C$ be the projection map. Obviously, p is surjective, and all its fibers are connected, since isomorphic to \mathbb{C}^* . Therefore, p is admissible (in the sense of §4.2), and the induced homomorphism, $p^*: H^1(C, \mathbb{C}) \rightarrow H^1(X^*, \mathbb{C})$, is an isomorphism.

By Theorem 4.4(1), the space $H^1(M, \mathbb{C})$ is not 1-isotropic. Thus, $H^1(X^*, \mathbb{C}) = p^*H^1(C, \mathbb{C})$ is also non-1-isotropic. In view of Theorem 4.2(2), the group $G = \pi_1(X^*)$ cannot be 1-formal. \square

Using the above proof, we can produce examples of smooth affine surfaces whose fundamental groups are not 1-formal. With notation as above, assume $g \geq 1$.

Proposition 7.2. *Let $f: X \rightarrow \mathbb{C}$ be a non-zero regular function with $f(0) = 0$. Suppose the curve $V(f) = \{x \in X \mid f(x) = 0\}$ intersects each fiber of the projection $p: X^* \rightarrow C$ in a finite number of points. Then $U = X \setminus V(f)$ is a smooth affine surface whose fundamental group is not 1-formal.*

Proof. Let $A(X)$ be the coordinate ring of X . The coordinate ring of U is $A(X)_f$, the localization of $A(X)$ with respect to the multiplicative system $\{f^k\}_{k \geq 1}$. Let $q: U \rightarrow C$ be the restriction of p to U , and let $\iota: U \hookrightarrow X^*$ be the inclusion map.

Assume $\pi_1(U)$ is 1-formal. Then, as above, Theorem 4.2(2) implies that the subspace $q^*(H^1(C, \mathbb{C})) = \iota^*(H^1(X^*, \mathbb{C}))$ is 1-isotropic. On the other hand, by the aforementioned result of Sullivan [39], the cup-product is trivial on $H^1(X^*, \mathbb{C})$. This contradiction ends the proof. \square

Here is an explicit family of examples.

Example 7.3. Let X be the surface in \mathbb{C}^3 given by the equation $x^d + y^d + z^d = 0$, with $d \geq 3$. Choose

$$(10) \quad f = x + y^2 + z^3 \in A(X) = \mathbb{C}[x, y, z]/(x^d + y^d + z^d).$$

It is readily verified that $V(f) \cap p^{-1}(c)$ is finite, for each $c \in C$. Thus, $U = X \setminus V(f)$ is a smooth affine surface with $\pi_1(U)$ not 1-formal.

8. TRANSLATED TORI IN CHARACTERISTIC VARIETIES

As in the previous section, let X be an affine surface with a good \mathbb{C}^* -action and isolated singularity at 0, $X^* = X \setminus \{0\}$, and $p: X^* \rightarrow C$ the corresponding projection onto a smooth projective curve.

Assume C has genus $g \geq 1$. We then know from Proposition 7.1(2) that the group $G = \pi_1(X^*)$ is not 1-formal. Of course, this prevents us from using Theorem 3.1 in computing the characteristic variety $V_1(G)$. Nevertheless, such a computation can still be carried out in this situation, using different techniques.

8.1. Pontrjagin duality. First, we need some additional notation. Let \mathcal{T} be the torsion part of the homology group $H_1(X^*, \mathbb{Z})$. Since $p^*: H^1(C, \mathbb{C}) \rightarrow H^1(X^*, \mathbb{C})$ is an isomorphism, the group \mathcal{T} is the kernel of $p_*: H_1(X^*, \mathbb{Z}) \rightarrow H_1(C, \mathbb{Z})$. Let $i_*: H_1(\mathbb{C}^*, \mathbb{Z}) \rightarrow H_1(X^*, \mathbb{Z})$ be the morphism induced by the inclusion of a generic fiber of p into X^* . It is clear that the image of i_* is a cyclic subgroup in \mathcal{T} , that is,

$$(11) \quad \text{im}(i_*) = \langle h \rangle, \text{ for some } h \in \mathcal{T}.$$

The direct sum decomposition $H_1(X^*, \mathbb{Z}) = \mathbb{Z}^{2g} \oplus \mathcal{T}$ yields a corresponding decomposition of the character group $\mathbb{T}(G) = \text{Hom}(G, \mathbb{C}^*)$,

$$(12) \quad \mathbb{T}(G) = (\mathbb{C}^*)^{2g} \times \hat{\mathcal{T}},$$

where $\hat{\mathcal{T}} = \text{Hom}(\mathcal{T}, \mathbb{C}^*)$ is the Pontrjagin dual of \mathcal{T} . In this decomposition, the identity component $\mathbb{T}^0(G)$ corresponds to the algebraic torus $(\mathbb{C}^*)^{2g} \times 1$.

Following [10], let us associate to the admissible map $p: X^* \rightarrow C$ the finite abelian group

$$(13) \quad T(p) = \mathcal{T}/\langle h \rangle.$$

The Pontrjagin dual $\hat{T}(p)$ may be regarded as a subgroup of $\hat{\mathcal{T}}$, and hence of $\mathbb{T}(G)$. Plainly, the index $[\hat{\mathcal{T}} : \hat{T}(p)]$ equals $\text{ord}(h)$, the order of the element h in \mathcal{T} .

8.2. Parametrizing translated components. We are now ready to state the main result of this section.

Proposition 8.1. *The positive-dimensional irreducible components of $V_1(X^*)$ consist of all the translates of the identity component $\mathbb{T}^0(G)$ by the elements of $\hat{T}(p)$ if $g > 1$, and by the elements of $\hat{T}(p) \setminus \{1\}$ if $g = 1$.*

Proof. As Arapura showed in [1], any positive-dimensional, irreducible component W of $V_1(X^*)$ corresponds to a surjective morphism $f: X^* \rightarrow S$ with connected generic fiber, where S is a smooth curve with $\chi(S) \leq 0$. As shown in [10], the components associated to such a morphism f are parametrized by $\hat{T}(f)$ when $\chi(S) < 0$, and by $\hat{T}(f) \setminus \{1\}$ when $\chi(S) = 0$. It thus remains to show that $p: X^* \rightarrow C$ is the only morphism with the properties of f above, up to isomorphisms of C .

Assume first that S is a curve of genus $h > 0$, i.e., the projective smooth closure \bar{S} of S has genus $h > 0$. Then each fiber of p , being isomorphic to \mathbb{C}^* , is mapped to a point by f (the only morphisms $\mathbb{C}\mathbb{P}^1 \rightarrow \bar{S}$ are the constant ones!). We may then factor f as $X^* \xrightarrow{p} C \xrightarrow{q} S$. This shows that $S = \bar{S}$. Moreover, since the generic fibers of f are connected, the morphism $q: C \rightarrow S$ has degree 1, i.e., it is an isomorphism.

Assume now S is a rational curve. Since $\chi(S) \leq 0$, the curve S is obtained from $\mathbb{C}\mathbb{P}^1$ by deleting at least 2 points. Hence, the mixed Hodge structure on $H^1(S, \mathbb{Q})$ is pure of weight 2. On the other hand, the mixed Hodge structure on $H^1(X^*, \mathbb{Q})$ is pure of weight 1, since $p^*: H^1(C, \mathbb{Q}) \rightarrow H^1(X^*, \mathbb{Q})$ is an isomorphism. Hence, $f^*: H^1(S, \mathbb{Q}) \rightarrow H^1(X^*, \mathbb{Q})$ cannot be injective, a contradiction. \square

8.3. Non-formality and the tangent cone formula. Comparing the conclusions of Propositions 7.1 and 8.1 reveals a rather subtle phenomenon: the tangent cone formula from Theorem 3.1 may hold, even when the group G is not 1-formal.

To see why this is the case, let us first compute the analytic germs at the origin for the cohomology jumping loci of $X^* = X \setminus \{0\}$. By Proposition 8.1,

$$(14) \quad (V_1(X^*), 1) = \begin{cases} (\{1\}, 1) & \text{if } g = 1, \\ ((\mathbb{C}^*)^{2g}, 1) & \text{if } g > 1. \end{cases}$$

On the other hand, X^* is homotopy equivalent to the singularity link M . It follows from Theorem 4.4(2) that

$$(15) \quad R_1(X^*) = H^1(X^*, \mathbb{C}) = \mathbb{C}^{2g}, \quad \text{for all } g \geq 1.$$

Corollary 8.2. *Let $G = \pi_1(X^*)$ as above, and assume $g > 1$. Then G is not 1-formal, yet the tangent cone formula $TC_1(V_1(G)) = R_1(G)$ holds.*

Proof. Since $g \geq 1$, Proposition 7.1(2) insures that G is not 1-formal. Using formulas (14) and (15), we see that the analytic isomorphism from Theorem 3.1, $\exp: (R_1(G), 0) \xrightarrow{\cong} (V_1(G), 1)$, holds. In particular, $TC_1(V_1(G)) = R_1(G)$. \square

8.4. Seifert invariants. As above, let M be the 3-manifold associated to the quasi-homogenous singularity $(X, 0)$. The S^1 -equivariant diffeomorphism type of M is determined by the following Seifert invariants associated to the projection $p|_M : M \rightarrow C$:

- The exceptional orbit data, $((\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m))$, with $\alpha_i, \beta_i \in \mathbb{Z}$, $\alpha_i > 1$, and $\gcd(\alpha_i, \beta_i) = 1$.
- The genus $g = g(C)$ of the base curve $C = M/S^1$, with $g \geq 0$.
- The Euler number e of the Seifert fibration, with $e \in \mathbb{Q}$, and $e < 0$.

For full details, see [28, 29, 30, 31].

Since $e \neq 0$, we have $H_1(M, \mathbb{Z}) = \mathbb{Z}^{2g} \oplus \mathcal{T}$, with \mathcal{T} finite, as in §8.1. Let $h \in \mathcal{T}$ be the element defined by (11). Using the proof of Theorem 4.1 from [30], we compute

$$(16) \quad |\mathcal{T}| = \alpha_1 \cdots \alpha_m \cdot |e|, \quad \text{ord}(h) = \text{lcm}(\alpha_1, \dots, \alpha_m) \cdot |e|.$$

It follows that the group $T(p) = \mathcal{T}/\langle h \rangle$ from (13) has order

$$(17) \quad \alpha = \alpha_1 \cdots \alpha_m / \text{lcm}(\alpha_1, \dots, \alpha_m).$$

Remark 8.3. The morphism $p: X^* \rightarrow C$ has exactly m multiple fibers, each isomorphic to \mathbb{C}^* , and with multiplicities $\alpha_1, \dots, \alpha_m$. Thus, formula (17) for the order of $T(p)$ may alternatively be obtained from [10, Theorem 5.3(i)].

Let $G = \pi_1(M)$. Recall that the character variety $\mathbb{T}(G)$ can be identified with $\mathbb{T}^0(G) \times \hat{\mathcal{T}}$, a disjoint union of $|\mathcal{T}|$ copies of the algebraic torus $\mathbb{T}^0(G) = (\mathbb{C}^*)^{2g}$. Proposition 8.1 now yields the following corollary.

Corollary 8.4. *The positive-dimensional components of $V_1(M)$ are: $\mathbb{T}^0(G)$ if $g > 1$, and $\alpha - 1$ translated copies of $\mathbb{T}^0(G)$, for any $g \geq 1$.*

In particular, $V_1(M)$ contains positive-dimensional components not passing through the origin if and only if $\alpha \neq 1$.

Example 8.5. When $M \rightarrow C$ is a bona-fide circle bundle, with Euler class $e \in \mathbb{Z}$, we may take $m = 0$, in which case $|\mathcal{T}| = \text{ord}(h) = |e|$. Thus, $V_1(M)$ has positive-dimensional components only if $g > 1$ —and then the only such component is $\mathbb{T}^0(G)$.

8.5. Brieskorn manifolds. Let (a_1, \dots, a_n) be an n -tuple of integers, with $a_j \geq 2$. Consider the variety X in \mathbb{C}^n defined by the equations $c_{j1}x_1^{a_1} + \dots + c_{jn}x_n^{a_n} = 0$, for $1 \leq j \leq n-2$. Assuming the matrix of coefficients (c_{jk}) has all maximal minors non-zero, X is a quasi-homogeneous surface, whose singularity link at 0 is the well-known Brieskorn manifold $M = \Sigma(a_1, \dots, a_n)$.

Set $\ell = \text{lcm}(a_1, \dots, a_n)$, $\ell_j = \text{lcm}(a_1, \dots, \widehat{a_j}, \dots, a_n)$, and $a = a_1 \cdots a_n$. Computations from [30, 28] show that the Seifert invariants of M are:

- $(s_1(\alpha_1, \beta_1), \dots, s_n(\alpha_n, \beta_n))$, with $\alpha_j = \ell/\ell_j$, $\beta_j \ell \equiv a_j \pmod{\alpha_j}$, and $s_j = a/(a_j \ell_j)$, where $s_j(\alpha_j, \beta_j)$ signifies (α_j, β_j) repeated s_j times, unless $\alpha_j = 1$, in which case $s_j(\alpha_j, \beta_j)$ is to be removed from the list.
- $g = \frac{1}{2}(2 + (n-2)a/\ell - \sum_{j=1}^n s_j)$.
- $e = -a/\ell^2$.

The case $n = 3$ was studied in detail by Milnor in [26]. The simplest situation is when $\ell_1 = \ell_2 = \ell_3 = \ell$, in which case $\alpha_j = 1$, for all j . Therefore, $M = \Sigma(a_1, a_2, a_3)$ is a smooth circle bundle over Σ_g , with Euler number e as above, and the positive-dimensional components of $V_1(M)$ are as described in Example 8.5.

Example 8.6. As noted by Milnor [26], the manifolds $M_1 = \Sigma(2, 3, 6)$, $M_2 = \Sigma(2, 4, 4)$, and $M_3 = \Sigma(3, 3, 3)$ are all nilmanifolds—in fact, circle bundles over the torus, with Euler number 1, 2, and 3, respectively. We know from the above that $V_1(M_i)$ has no positive-dimensional components. More is true, though: as shown in [23, Theorem 1.1], $V_1(M_i) = \{1\}$. In other words, there are no isolated points in $V_1(M_i)$, either, besides the identity.

In general, though, the first characteristic variety of a Brieskorn manifold will contain translated components.

Example 8.7. The manifold $M = \Sigma(3, 3, 6)$ has Seifert invariants $((2, 1), (2, 1), (2, 1))$, $g = 1$, and $e = -3/2$. By Corollary 8.4, $V_1(M)$ has 3 positive-dimensional irreducible components, all of dimension 2, none of which passes through the identity.

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LABORATOIRE J.A. DIEUDONNÉ, UMR DU CNRS 6621, UNIVERSITÉ DE NICE–SOPHIA ANTIPOLIS,
 PARC VALROSE, 06108 NICE CEDEX 02, FRANCE
E-mail address: dimca@math.unice.fr

INSTITUTE OF MATHEMATICS SIMION STOILOW, P.O. BOX 1-764, RO-014700 BUCHAREST, ROMANIA
E-mail address: Stefan.Papadima@imar.ro

DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MA 02115, USA
E-mail address: a.suciu@neu.edu