Homology of jet groups

Emmanuel Dror Farjoun

Institute of Mathematics, The Hebrew University of Jerusalem, Israel

Solomon M. Jekel

Department of Mathematics, Northeastern University, Boston, MA 02115, USA

Alexander I. Suciu [1]

Department of Mathematics, Northeastern University, Boston, MA 02115, USA

1 Jet groups

In this paper we compute the second homology of the discrete jet groups. Let $\mathbf{R}$ be the additive group of real numbers and $\mathbf{R}^+$ the multiplicative group of positive reals. The n^{th} *jet group* $J_n = \{rx + a_2x^2 + \cdots + a_nx^n \mid r \in \mathbf{R}^+, a_i \in \mathbf{R}\}$ is the group, under composition followed by truncation, of invertible, orientation-preserving real n-jets at 0. Consider the homomorphism $D : J_n \to \mathbf{R}^+$ obtained by projecting onto the first coefficient, i.e. Df = first derivative of f at 0. Every jet with slope not equal to 1 is conjugate to its linear part. It follows there is a split exact sequence

$$1 \to J'_n \longrightarrow J_n \xrightarrow{D} \mathbf{R}^+ \to 1, \qquad (1)$$

with splitting $\sigma : \mathbf{R}^+ \to J_n, \sigma(r) = rx$. The map $D_* : H_k(J_n) \to H_k(\mathbf{R}^+)$ is an epimorphism, since it admits σ_* as right inverse. We conjecture that in fact D_* is an isomorphism, for all $k \geq 0$. It follows from (1) that $D_* : H_1(J_n) \to H_1(\mathbf{R}^+)$ is an isomorphism. The main result of this paper is:

Theorem 1.1 *The map $D_* : H_2(J_n) \to H_2(\mathbf{R}^+)$ is an isomorphism.*

The structure of $H_2(\mathbf{R}^+)$ is easy to describe. For an abelian group A, $H_2(A)$ is naturally isomorphic to $(A \otimes_{\mathbf{Z}} A)/\Delta$, where Δ = diagonal (Miller [7], Brown [1, pp. 121–127]. Now $\mathbf{R}^+$ is isomorphic as an abelian group to $\mathbf{R}$, which is an

[1] Partially supported by a Northeastern University Junior Research Fellowship.

Article published in J. Pure and Applied Algebra 102 (1995) 17–24

uncountable direct sum of $\mathbf{Q}$'s; thus $H_2(\mathbf{R}^+)$ is also an uncountable direct sum of $\mathbf{Q}$'s.

We are ultimately interested in the second homology group of the discrete group G_0^ω of convergent invertible series at the origin of $\mathbf{R}$, for this is a crucial term in the classification of cobordism classes of real analytic Γ-structures on surfaces [4]. A next step towards computing $H_2(G_0^\omega)$ would be the determination of $H_2(J_\infty)$, where $J_\infty = \varprojlim J_n$ is the group of formal invertible series at 0. It may be possible that there are elements in $H_2(J_\infty)$ other than those in $H_2(\mathbf{R}^+)$. For more motivation and discussion, as well as another proof of the theorem for $n \leq 3$, see [6].

2 An E^1 spectral sequence converging to $H_*(G)$

Let G stand for an arbitrary group. We define $H_*(G)$, the integral homology of G as a discrete group, to be $H_*(BG;\mathbf{Z})$, where BG is the classifying space of G (see [1], [8]). Let us recall the sequence of constructions leading to $H_*(BG)$. The space BG is the geometric realization of the simplicial nerve of G, NG : $NG^{(0)} = \{1\}, NG^{(n)} = G \times \cdots \times G$ (n times). The face maps d_i are defined by $d_0(g_1, \ldots, g_n) = (g_2, \ldots, g_n)$, $d_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i g_{i+1}, \ldots, g_n)$, for $0 < i < n$, and $d_n(g_1, \ldots, g_n) = (g_1, \ldots, g_{n-1})$. Next one forms the chains on NG, $C_*(NG)$, with differential $d = \sum(-1)^i d_i$; the homology of G is the homology of this chain complex. The complex $C_*(NG)$ is often referred to as the bar construction on G. The space BG, of which it computes the homology, is a $K(G, 1)$.

Consider now a short exact sequence of groups

$$0 \to A \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1, \tag{2}$$

where A is abelian and A is normal in G. The quotient Q acts on A by conjugation, $h \cdot a = g^{-1} a g$, where $a \in A$ and g is any element of G so that $\pi(g) = h$ (see [1], pp. 86–87). The group Q then acts on the integral homology of A. Let us also use the notation $h \cdot \alpha$ for this action ($h \in Q, \alpha \in H_k(A)$). The context will always make the domain of the action clear.

Theorem 2.1 (a) *There is a spectral sequence with* $E^1_{p,q} = \bigoplus_{Q^p} H_q(A)$ *converging to* $H_{p+q}(G)$.

We next describe the differentials $d^1_{p,q}, p \geq 1, q \geq 0$. Let us write $\alpha(h_1, \ldots, h_p)$ for the element of $E^1_{p,q}$ which has $\alpha \in H_q(A)$ in the summand corresponding to $(h_1, \ldots, h_p) \in Q^p$ and all other components 0. Define face maps $\partial_i : E^1_{p,q} \to E^1_{p-1,q}$, $i = 0, \ldots, p$ by

$$\begin{cases} \partial_0(\alpha(h_1, \ldots, h_p)) = h_1 \cdot \alpha(h_1^{-1}h_2, \ldots, h_1^{-1}h_p) \\ \partial_i(\alpha(h_1, \ldots, h_p)) = h_1 \cdot \alpha(h_1, \ldots, \hat{h}_i, \ldots, h_p). \end{cases}$$

(For $p = 1$, this should be interpreted as $\partial_0(\alpha(h)) = h \cdot \alpha, \partial_1(\alpha(h)) = \alpha$.) Note, only ∂_0 involves the Q-action.

Theorem 2.1 (b) *The differential $d^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$ is given by*

$$d^1_{p,q} = \sum_{i=0}^{p} (-1)^i \partial_i.$$

Notice that the complex $E^1_{*,0}$ is the bar construction on Q.

Theorem 2.1 is Theorem 2 of [5] in the special case when the group G acts on the cosets $G/A = Q$ by left multiplication. The more general spectral sequence associated to a group acting on a set has its origins in the work of Ehresmann. Although classical, it is less well-known than the Hochschild-Serre spectral sequence. We choose to use the former because the explicit formulas for the E^1 differentials, which we require, are easy to compute. In section 4 we recall the constructions of [5], and use them to derive the above spectral sequence. First though we use it to furnish a proof of Theorem 1.1.

3 Calculation of $H_2(J_n)$

Given $n > k \geq 1$, let $p_k : J_n \to J_k$ be the projection map. Its kernel is $J_{n,k} = \{x + a_{k+1}x^k + \cdots + a_n x^n\}$. Notice that J_1 is $\mathbf{R}^+$, p_1 is the map D from (1), and so $J_{n,1} = J'_n$. Thus in fact $J_{n,k}$ is a subgroup of J'_n. The group $J_{n,n-1}$ is isomorphic to $\mathbf{R}$. We thus have the exact sequence

$$0 \to \mathbf{R} \longrightarrow J_n \xrightarrow{p_{n-1}} J_{n-1} \to 1. \tag{3}$$

In this extension, the group J_{n-1} acts on $\mathbf{R}$ via

$$g \cdot a = (Dg)^{n-1} a, \tag{4}$$

that is, a is multiplied by the first derivative of $g \in J_{n-1}$ raised to the $(n-1)$-st power.

We now start the proof of Theorem 1.1. We will show by induction on n that $D_* : H_2(J_n) \to H_2(\mathbf{R}^+)$ is an isomorphism. For $n = 1$ this is clear, as D identifies J_1 with $\mathbf{R}^+$. Assume $D_* : H_2(J_{n-1}) \to H_2(\mathbf{R}^+)$ is an isomorphism, and consider the spectral sequence of section 2 for the extension (3). To prove the theorem it suffices to show that the terms $E^2_{0,2}$ and $E^2_{1,1}$ vanish. For then $(p_{n-1})_* : H_2(J_n) \longrightarrow H_2(J_{n-1})$ is an isomorphism and by induction $D_* : H_2(J_n) \longrightarrow H_2(\mathbf{R}^+)$ is one also.

Lemma 3.1 $E^2_{0,2} = 0$.

Proof. The relevant term is $H_2(\mathbf{R}) \stackrel{d}{\longleftarrow} \bigoplus_{J_{n-1}} H_2(\mathbf{R})$, where, by Theorem 2.1 and (4), $d(\alpha(g)) = (Dg)^{n-1}\alpha - \alpha$, for $\alpha \in H_2(\mathbf{R}), g \in J_{n-1}$. Under the isomorphism $H_2(\mathbf{R}) \cong (\mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R})/\Delta$, the map d is given by

$$a \otimes b(g) \mapsto (Dg)^{n-1}a \otimes (Dg)^{n-1}b - a \otimes b. \tag{5}$$

Notice that $H_2(\mathbf{R})$ is a real vector space and d is an $\mathbf{R}$-linear map.

Let $a \otimes b$ be an arbitrary element of $E^2_{0,2}$. Pick g to be the $(n-1)$-st root of 2 in $\mathbf{R}^+ \subset J_n$. Then, by (5), (modulo the image of d,) the following equalities hold: $a \otimes b = 2a \otimes 2b = 4[a \otimes b]$, or $3[a \otimes b] = 0$. Thus $a \otimes b = 0$, and so $E^2_{0,2} = 0$, as claimed.

Lemma 3.2 $E^2_{1,1} = 0$.

Proof. We consider the chain complex $E_* = (E_p, d_p)$, where

$$E_p = E^1_{p,1} = \bigoplus_{J^p_{n-1}} \mathbf{R}$$
$$d_p = d^1_{p,1}$$

The terms relevant to our calculation are given explicitly as

$$\mathbf{R} \stackrel{d_1}{\longleftarrow} \bigoplus_{J_{n-1}} \mathbf{R} \stackrel{d_2}{\longleftarrow} \bigoplus_{J^2_{n-1}} \mathbf{R},$$

with

$$d_1(a(g)) = (Dg)^{n-1}a - a,$$
$$d_2(a(f,g)) = [(Df)^{n-1}a](f^{-1}g) - a(g) + a(f).$$

We wish to show the real vector space $E^2_{1,1} = \ker d_1 / \operatorname{im} d_2$ is 0. We achieve this in a sequence of lemmas.

There is a subcomplex $\hat{E}_*$ of E_* spanned by the linear jets, given by

$$\hat{E}_p = \bigoplus_{(\mathbf{R}^+)^p} \mathbf{R}.$$

Lemma 3.3 *$\hat{E}_*$ is an acyclic chain complex.*

Proof. Let $a = 2^{\frac{1}{n-1}}$. We define a chain contraction $T = T_p : \hat{E}_p \to \hat{E}_{p+1}$ from the identity to 0 as follows:

$$\begin{aligned} T_0(1) &= a, \\ T_p(g_1, \ldots, g_p) &= (a, ag_1, \ldots, ag_p) - (g_1, ag_1, \ldots, ag_p) + (g_1, g_2, ag_2, \ldots, ag_p) - \\ &\quad (g_1, g_2, g_3, ag_3, \ldots, ag_p) + \cdots + (-1)^p (g_1, g_2, \ldots, g_{p-1}, ag_p), \end{aligned}$$

and extend by linearity. Then $\partial T_p + T_{p-1}\partial =$ identity. We verify this for $p = 1$ only, which is the term relevant to our calculation. The formulas in this case are

$$\begin{aligned} T_0(1) &= (a), \\ T_1(g) &= (a, ag) - (g, ag), \\ \partial T_1(g) &= 2(g) - (ag) + (a) - g^{n-1}(a) + (ag) - (g) = (g) - g^{n-1}(a) + (a), \\ T_0\partial(g) &= T_0(g^{n-1}(1) - (1)) = g^{n-1}(a) - (a). \end{aligned}$$

Note that for $n = 2$, $\hat{E}_* = E_*$. Thus Lemma 3.2 is proved in this case. For the rest of this section we will assume $n > 2$.

Remark The chain contraction T can be extended to all the horizontal chain complexes in the E^1 term. We don't do this here, but this would imply that $H_k(J_2) \cong H_k(\mathbf{R}^+)$ for all $k \geq 0$, which is known (see Greenberg [2] and the discussion in [6]).

Lemma 3.4 *$E^2_{1,1}$ is generated as a vector space by elements of $\ker d_1$ of the form (f) with $f \in J'_{n-1}$.*

Proof. Let $z = \sum a_j(g_j)$ be in $\ker d_1$. Then $z = \sum a_j[(g_j) - (Dg_j)] + \sum a_j(Dg_j)$. Each $[(g_j) - (Dg_j)]$ is in $\ker d_1$ and hence $\sum a_j(Dg_j)$ is also. By Lemma 3.3,

$\sum a_j(Dg_j)$ belongs to $\operatorname{im} d_2$. This shows $E^2_{1,1}$ is generated by elements of $\ker d_1$ of the form $(g) - (Dg)$.

Consider now the generator (Dg, g) of $E^1_{2,1}$. We have

$$d_2(Dg, g) = (Dg)^{n-1}((Dg)^{-1}g) - (g) + (Dg).$$

The jet $(Dg)^{-1}g$ has slope 1, so the calculation shows that, modulo $\operatorname{im} d_2$, $(g) - (Dg) = a(f)$, with $Df = 1$. This finishes the proof.

To complete the proof of Lemma 3.2 we need to show that any (f) as in Lemma 3.4 is in $\operatorname{im} d_2$. We will write $(f) \sim (g)$ if $(f) - (g) \in \operatorname{im} d_2$. Notice that if $g, h \in J'_{n-1}$, then $d_2((g, gh)) = (h) - (gh) + (g)$. Thus $(gh) \sim (g) + (h)$.

Lemma 3.5 *Let $f \in J'_{n-1}$, and $a \in R^+ \subset J_{n-1}$. Then $(f^2) \sim 2a^{n-1}(a^{-1}fa)$.*

Proof. Direct computation gives

$$\begin{aligned} d_2((a, fa) + (f^{-1}, a) - (f, 1) - (1, 1)) &= a^{n-1}(a^{-1}fa) - (f), \\ d_2((f^{-1}, f) - (f^{-1}, 1) - (1, 1)) &= (f^2) - 2(f). \end{aligned}$$

Corollary 3.6 *If $f \in J_{n-1,n-2}$ then $(f) \sim 0$.*

Proof. Write $f = x + a_{n-1}x^{n-1}$. Then $f^2 = x + 2a_{n-1}x^{n-1}$. Applying the previous lemma with $a = 2^{\frac{1}{n-2}}$ yields

$$x + 2a_{n-1}x^{n-1} = 2^{\frac{n-1}{n-2}+1}(x + 2a_{n-1}x^{n-1})$$

This implies $(f)^2 \sim 0$ and hence $(f) \sim 0$.

Lemma 3.7 *If $f \in J'_{n-1}$ then $(f) \sim 0$.*

Proof. We will show by induction on k that if $f \in J_{n-1,n-k}$, and $2 \leq k \leq n-1$, then $(f) \sim 0$. The lemma will then follow because $J_{n-1,1} = J'_{n-1}$. Corollary 3.6 is the step $k = 2$ of the induction. Assume $(f) \sim 0$ for all $f \in J_{n-1,n-k}$. Let g be an arbitrary element of $J_{n-1,n-(k+1)}$. Then g can be written as hf, where $h = x + a_{n-k}x^{n-k}$ and $f \in J_{n-1,n-k}$. We wish to show that $(g) \sim 0$. By the remark preceding Lemma 3.5 and the induction hypothesis, it suffices to show that $(h) \sim 0$.

Applying Lemma 3.5 with $a = 2^{\frac{1}{n-k}}$ yields

$$(h^2) = 2^{\frac{n-1}{n-k}+1}(x + 2a_{n-k}x^{n-k})$$

On the other hand, $h^2 = (x + 2a_{n-k}x^{n-k})f_0$, for some $f_0 \in J_{n-1,n-k}$. Thus

$$(h^2) \sim (x + 2a_{n-k}x^{n-k}).$$

Hence $(x + 2a_{n-k}x^{n-k}) \sim 0$ and so $(h) \sim 0$.

This finishes the proof of Lemma 3.2 and thus the proof of Theorem 1.1. Notice that, instead of working over the reals, one could work over any subfield of **R** that contains all the roots of some positive number $r \neq 1$. Then r would play the role of 2 in the above Lemmas.

4 Derivation of the spectral sequence

We now construct the E^1-spectral sequence of Theorem 2.1. We start by recalling some facts about discrete groupoids (see Higgins [3]).

A groupoid is a small category Γ in which every morphism is an isomorphism. We will identify the objects of with the identity morphisms. Given a discrete groupoid, Γ define an equivalence relation $\approx$ on Objects(Γ): $x \approx y$ if there is a morphism from x to y. Clearly $\pi_0(B\Gamma)$ is in one-to-one correspondence with Objects$(\Gamma)/\approx$. Choose one object α in each equivalence class. The set Θ is *a set of base points* for Γ. The isotropy group of the base point α is the group π_α of all morphisms in Γ whose source and target is α. It is well known (and easy to prove) that

$$B\Gamma = \coprod_{\alpha\in\Theta} K(\pi_\alpha, 1). \tag{6}$$

Next let $F : \Gamma \to \Gamma'$ be a functor of groupoids, and let Θ and Θ' be sets of base points for Γ and Γ', respectively. For each object x of Γ pick once and for all a morphism $\rho(x)$ from the base point of the component containing x to x. The set $\{\rho(x)\}$ is a set of base paths for Γ'. Then F induces a homomorphism $F_\sharp : \pi_\alpha \to \pi_{\alpha'}$ where α' is the base point of the component containing $F(\alpha)$. Namely $F_\sharp(m) = \rho(F(m))^{-1} \circ F(m) \circ \rho(F(m))$, for $m \in \pi_\alpha$.

Now consider the extension $0 \to A \to G \to Q \to 1$ of section 2. For each $p \geq 0$ define a discrete groupoid $\{G/A\}_p$ as follows: Objects$\{G/A\}_p = Q^{p+1}$, Morphisms$\{G/A\}_p = G \times Q^{p+1}$.

A typical object is a $(p+1)$-tuple of cosets $(h_0A, \ldots, h_pA)$. As a set of base points we may take all $(p+1)$-tuples of the form $(A, h_1A, \ldots, h_pA)$. Then $\pi_0\{G/A\}_p$ is in one-to-one correspondence with $(G/A)^p = Q^p$. Furthermore, the isotropy group of each base point is simply A.

The groupoids $\{G/A\}_p$ fit together to form a simplicial groupoid $\{G/A\}_*$, with face maps δ_i given by

$$\delta_i(gA, h_0A, \ldots, h_pA) = (gA, h_0A, \ldots, \widehat{h_iA}, \ldots, h_pA).$$

Now form a bisimplicial set $\{G/A\}_{*,*}$ by extending by nerves in the vertical direction. Computing vertical homology yields in the (p,q)-th place the q-th homology of the discrete groupoid $\{G/A\}_p$. Then standard considerations lead to a double complex, and a spectral sequence converging to BG (see [5]).

Theorem 4.1 *There is a spectral sequence with $E^1_{p,q} = H_q\{G/A\}_p$ converging to $H_{p+q}(G)$. The differential $d^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$ is given by $d^1_{p,q} = \sum_{i=0}^{p}(-1)^i\delta_i$*

To derive the spectral sequence of Theorem 2.1 from that of Theorem 4.1 we will identify their respective E^1 terms and differentials. The $E^1_{p,q}$ term of Theorem 4.1 is equal to $H_q(B\{G/A\}_p)$, by definition, which in turn is equal to $\bigoplus_\alpha H_q(\pi_\alpha)$, by (6), which is $\bigoplus_{Q^p} H_q(A)$, by our earlier observations, which is $E^1_{p,q}$, by construction.

It remains to show that under these identifications the δ_i face maps of Theorem 4.1 carry over in homology to the ∂_i face maps of Theorem 2.1. For $i \geq 1$, δ_i is a functor which preserves the chosen sets of base points. Therefore δ_i induces the identity on isotropy groups. Obviously ∂_i is the homomorphism induced on vertical homology by δ_i.

On the other hand, when $i = 0$, δ_0 maps the base point $\alpha = (A, h_1A, \ldots, h_pA)$ to the component containing the object $\delta_0(\alpha) = (h_1A, \ldots, h_pA)$ The base point of this component is $(A, h_1^{-1}h_2A, \ldots, h_1^{-1}h_pA)$ and a base path is

$$\rho(\delta_0(\alpha) = (h_1, A, h_1^{-1}h_2A, \ldots, h_1^{-1}h_pA).$$

Therefore δ_0 induces, on $B\{G/A\}_p$, $(h_1, \ldots, h_p) \to (h_1^{-1}h_2, \ldots, h_1^{-1}h_p)$, and the homomorphism $(\delta_0)_\sharp$ induced on the isotropy group A is $m \to h_1^{-1}mh_1$. Clearly then ∂_0 is the homomorphism induced on vertical homology by δ_0.

This finishes the identification of the differentials in the two spectral sequences, and thus the proof of Theorem 2.1.

Note added in proof. The conjecture in section 1 has been verified by P. Dartnell [On the homology of groups of jets, *J. Pure Appl. Alg.* **92** (1994), 109–121], and further generalized by W. Dwyer, S. Jekel, and A. Suciu [Homology isomorphisms between algebraic groups made discrete, *Bull. London Math. Soc.* **25** (1993), 145–149].

References

[1] K.S. Brown, *Cohomology of Groups* (Springer, New York, 1982).

[2] P. Greenberg, Classifying spaces for foliations with isolated singularities, *Trans. Amer. Math. Soc.* **304** (1987), 417–429.

[3] P. Higgins, *Categories and Groupoids* (Van Nostrand-Reinhold Mathematical Series 32, London, 1971).

[4] S.M. Jekel, A spectral sequence for pseudogroups on **R**, *Trans. Amer. Math. Soc.* **333** (1992), 741–749.

[5] S.M. Jekel, On the second homology group of the discrete group of diffeomorphisms of the circle, *J. Pure Appl. Algebra* **59** (1989), 255–264.

[6] S.M. Jekel and A.I. Suciu, The second homology of discrete groups of jets, preprint.

[7] C. Miller, The second homology of a group; relations among commutators, *Proc. Amer. Math. Soc.* **3** (1952), 588–595.

[8] G.B. Segal, Classifying spaces and spectral sequences, *Publications I.H.E.S.* **34** (1968), 105–112.