GEOMETRIC AND HOMOLOGICAL FINITENESS IN FREE ABELIAN COVERS

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ABSTRACT. We describe some of the connections between the Bieri–Neumann–Strebel–Renz invariants, the Dwyer–Fried invariants, and the cohomology support loci of a space X. Under suitable hypotheses, the geometric and homological finiteness properties of regular, free abelian covers of X can be expressed in terms of the resonance varieties, extracted from the cohomology ring of X. In general, though, translated components in the characteristic varieties affect the answer. We illustrate this theory in the setting of toric complexes, as well as smooth, complex projective and quasi-projective varieties, with special emphasis on configuration spaces of Riemann surfaces and complements of hyperplane arrangements.

CONTENTS

1. INTRODUCTION

1.1. **Finiteness properties.** This investigation is motivated by two seminal papers that appeared in 1987: one by Bieri, Neumann, and Strebel [7], and the other by Dwyer and Fried [17]. Both papers dealt with certain finiteness properties of normal subgroups of a group (or regular covers of a space), under the assumption that the factor group (or the group of deck transformations) is free abelian.

In [7], Bieri, Neumann, and Strebel associate to every finitely generated group G a subset $\Sigma^1(G)$ of the unit sphere $S(G)$ in the real vector space $\mathrm{Hom}(G, \mathbb{R})$. This "geometric" invariant of the group G is cut out of the sphere by open cones, and is independent of a finite generating set for G. In [8], Bieri and Renz introduced a nested family of higher-order invariants, $\{\Sigma^i(G, \mathbb{Z})\}_{i \geq 1}$, which record the finiteness properties of normal subgroups of G with abelian quotients.

In a recent paper [21], Farber, Geoghegan and Schütz further extended these definitions. To each connected, finite-type CW-complex X, these authors assign a sequence of invariants, $\{\Sigma^i(X, \mathbb{Z})\}_{i \geq 1}$, living in the unit sphere $S(X) \subset H^1(X, \mathbb{R})$. The sphere $S(X)$ can be thought of as parametrizing all free abelian covers of X, while the Σ-invariants (which are again open subsets), keep track of the geometric finiteness properties of those covers.

Another tack was taken by Dwyer and Fried in [17]. Instead of looking at all free abelian covers of X at once, they fix the rank, say r, of the deck-transformation group, and view the resulting covers as being parametrized by the rational Grassmannian $\mathrm{Gr}_r(H^1(X, \mathbb{Q}))$. Inside this Grassmannian, they consider the subsets $\Omega_r^i(X)$, consisting of all covers for which the Betti numbers up to degree i are finite, and show how to determine these sets in terms of the support varieties of the relevant Alexander invariants of X. Unlike the Σ-invariants, though, the Ω-invariants need not be open subsets, see [17] and [38].

Our purpose in this note is to explore several connections between the geometric and homological invariants of a given space X, and use these connections to derive useful information about the rather mysterious Σ-invariants from concrete knowledge of the more accessible Ω-invariants.

1.2. **Characteristic varieties and Ω-sets.** Let $G = \pi_1(X, x_0)$ be the fundamental group of X, and let $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$ be the group of complex-valued characters on G, thought of as the parameter space for rank 1 local systems on X. The key tool for comparing the aforementioned invariants are the *characteristic varieties* $\mathcal{V}^i(X)$, consisting of those characters $\rho \in \widehat{G}$ for which $H_j(X, \mathbb{C}_\rho) \neq 0$, for some $j \leq i$.

Let X^{ab} be the universal abelian cover of X, with group of deck transformations G_{ab}. We may view each homology group $H_j(X^{\mathrm{ab}}, \mathbb{C})$ as a finitely generated module over the Noetherian ring $\mathbb{C}G_{\mathrm{ab}}$. As shown in [33], the variety $\mathcal{V}^i(X)$ is the support locus for the direct sum of these modules, up to degree i. It follows then from the work of Dwyer and Fried [17], as further reinterpreted in [33, 38], that $\Omega_r^i(X)$ consists of those r-planes P inside $H^1(X, \mathbb{Q})$ for which the algebraic torus $\exp(P \otimes \mathbb{C})$ intersects the variety $\mathcal{V}^i(X)$ in only finitely many points.

Let $\mathcal{W}^i(X)$ be the intersection of $\mathcal{V}^i(X)$ with the identity component of $\widehat{G}$, and let $\tau_1(\mathcal{W}^i(X))$ be the set of points $z \in H^1(X, \mathbb{C})$ such that $\exp(\lambda z)$ belongs to $\mathcal{W}^i(X)$, for all $\lambda \in \mathbb{C}$. As noted in [16], this set is a finite union of rationally defined subspaces. For each $r \geq 1$, we then have

$$\Omega_r^i(X) \subseteq \mathrm{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{W}^i(X))), \tag{1}$$

where $\tau_1^{\mathbb{Q}}$ denotes the rational points on τ_1, and $\sigma_r(V)$ denotes the variety of incident r-planes to a homogeneous subvariety $V \subset H^1(X, \mathbb{Q})$, see [38]. There are many classes of spaces for which inclusion (1) holds as equality—for instance, toric complexes, or, more generally, the "straight spaces" studied in [37]—but, in general, the inclusion is strict.

1.3. **Comparing the Ω-sets and the Σ-sets.** A similar inclusion holds for the BNSR-invariants. As shown in [21], the set $\Sigma^i(X, \mathbb{Z})$ consists of those elements $\chi \in S(X)$ for which the homology of X with coefficients in the Novikov-Sikorav completion $\widehat{\mathbb{Z}G}_{-\chi}$ vanishes, up to degree i. Using this interpretation, we showed in [33] that the following inclusion holds:

$$\Sigma^i(X, \mathbb{Z}) \subseteq S(X) \setminus S(\tau_1^{\mathbb{R}}(\mathcal{W}^i(X))), \tag{2}$$

where $\tau_1^{\mathbb{R}}$ denotes the real points on τ_1, and $S(V)$ denotes the intersection of $S(X)$ with a homogeneous subvariety $V \subset H^1(X, \mathbb{R})$. Again, there are several classes of spaces for which inclusion (2) holds as equality—for instance, nilmanifolds, or compact Kähler manifolds without elliptic pencils with multiple fibers—but, in general, the inclusion is strict.

Clearly, formulas (1) and (2) hint at a connection between the Dwyer–Fried invariants and the Bieri–Neumann–Strebel–Renz invariants of a space X. We establish an explicit connection here, by comparing the conditions insuring those inclusions hold as equalities. Our main results reads as follows.

Theorem 1.1. *Let X be a connected CW-complex with finite k-skeleton. Suppose that, for some $i \leq k$,*

$$\Sigma^i(X, \mathbb{Z}) = S(X) \setminus S(\tau_1^{\mathbb{R}}(\mathcal{W}^i(X))).$$

Then, for all $r \geq 1$,

$$\Omega_r^i(X) = \operatorname{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{W}^i(X))).$$

Simple examples show that, in general, the above implication cannot be reversed.

The main usefulness of Theorem 1.1 resides in the fact that it allows one to show that the Σ-invariants of a space X are smaller than the upper bound given by (2), once one finds certain components in the characteristic varieties of X (e.g., positive-dimensional translated subtori) insuring that the Ω-invariants are smaller than the upper bound given by (1).

1.4. **Formality, straightness, and resonance.** The above method can still be quite complicated, in that it requires computing cohomology with coefficients in rank 1 local systems on X. As noted in [33] and [37], though, in favorable situations the right-hand sides of (1) and (2) can be expressed in terms of ordinary cohomological data.

By definition, the i-th *resonance variety* of X, with coefficients in a field $\Bbbk$ of characteristic 0, is the set $\mathcal{R}^i(X, \Bbbk)$ of elements $a \in H^1(X, \Bbbk)$ for which the cochain complex whose terms are the cohomology groups $H^j(X, \Bbbk)$, and whose differentials are given by multiplication by a fails to be exact in some degree $j \leq i$. It is readily seen that each of these sets is a homogeneous subvariety of $H^1(X, \Bbbk)$.

If X is 1-formal (in the sense of rational homotopy theory), or, more generally, if X is locally 1-straight, then $\tau_1(\mathcal{W}^1(X)) = \operatorname{TC}_1(\mathcal{W}^1(X)) = \mathcal{R}^1(X)$. Thus, formulas (1) and (2) yield the following inclusions:

$$\Omega_r^1(X) \subseteq \operatorname{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r(\mathcal{R}^1(X, \mathbb{Q})), \tag{3}$$

$$\Sigma^1(X, \mathbb{Z}) \subseteq S(X) \setminus S(\mathcal{R}^1(X, \mathbb{R})). \tag{4}$$

If X is locally k-straight, then the analogue of (3) holds in degrees $i \leq k$, with equality if X is k-straight. In general, though, neither (3) nor (4) is an equality.

Applying now Theorem 1.1, we obtain the following corollary.

Corollary 1.2. *Let X be a locally 1-straight space (for instance, a 1-formal space).*
Suppose $\Sigma^1(X, \mathbb{Z}) = S(X) \setminus S(\mathcal{R}^1(X, \mathbb{R}))$. Then, for all $r \geq 1$,

$$\Omega_r^1(X) = \mathrm{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r(\mathcal{R}^1(X, \mathbb{Q})).$$

1.5. **Applications.** We illustrate the theory outlined above with several classes of examples, coming from toric topology and algebraic geometry, as well as the study of configuration spaces and hyperplane arrangements.

1.5.1. *Toric complexes.* Every simplicial complex L on n vertices determines a subcomplex T_L of the n-torus, with k-cells corresponding to the $(k-1)$-simplices of L. The fundamental group $\pi_1(T_L)$ is the right-angled Artin group G_L attached to the 1-skeleton of L, while a classifying space for G_L is the toric complex associated to the flag complex Δ_L.

It is known that all toric complexes are both straight and formal; their characteristic and resonance varieties were computed in [31], whereas the Σ-invariants of right-angled Artin groups were computed in [29, 10]. These computations, as well as work from [33, 37] show that $\Omega_r^1(T_L) = \sigma_r(\mathcal{R}^i(T_L, \mathbb{Q}))^{\complement}$ and $\Sigma^i(G_L, \mathbb{Z}) \subseteq S(\mathcal{R}^i(G_L, \mathbb{R}))^{\complement}$, though this last inclusion may be strict, unless a certain torsion-freeness assumption on the subcomplexes of Δ_L is satisfied.

1.5.2. *Quasi-projective varieties.* The basic structure of the characteristic varieties of smooth, complex quasi-projective varieties was determined by Arapura [1], building on work of Beauville, Green and Lazarsfeld, and others. In particular, if X is such a variety, then all the components of $\mathcal{W}^1(X)$ are torsion-translated subtori.

If X is also 1-formal (e.g., if X is compact), then inclusions (3) and (4) hold, but not always as equalities. For instance, if $\mathcal{W}^1(X)$ has a 1-dimensional component not passing through 1, and $\mathcal{R}^1(X, \mathbb{C})$ has no codimension-1 components, then, as shown in [37], inclusion (3) is strict for $r = 2$, and thus, by Corollary 1.2, inclusion (4) is also strict.

1.5.3. *Configuration spaces.* An interesting class of quasi-projective varieties is provided by the configuration spaces $X = F(\Sigma_g, n)$ of n ordered points on a closed Riemann surface of genus g. If $g = 1$ and $n \geq 3$, then the resonance variety $\mathcal{R}^1(X, \mathbb{C})$ is irreducible and non-linear, and so X is not 1-formal, by [16]. To illustrate the computation of the Ω-sets and Σ-sets in such a non-formal setting, we work out the details for $F(\Sigma_1, 3)$.

1.5.4. *Hyperplane arrangements.* Given a finite collection of hyperplanes, $\mathcal{A}$, in a complex vector space $\mathbb{C}^\ell$, the complement $X(\mathcal{A})$ is a smooth, quasi-projective variety; moreover, $X(\mathcal{A})$ is formal, locally straight, but not always straight. Arrangements of hyperplanes have been the main driving force behind the development of the theory of cohomology jump loci, and still provide a rich source of motivational examples for this theory. Much is known about the characteristic and resonance varieties of arrangement complements; in particular, $\mathcal{R}^1(X(\mathcal{A}), \mathbb{C})$ admits a purely combinatorial description, owing to the pioneering work of Falk [19], as sharpened by many others since then. We give here both lower bounds and upper bounds for the Ω-invariants and the Σ-invariants of arrangements.

In [37], we gave an example of an arrangement $\mathcal{A}$ for which the Dwyer–Fried set $\Omega_2^1(X(\mathcal{A}))$ is strictly contained in $\sigma_2(\mathcal{R}^1(X(\mathcal{A}), \mathbb{Q}))^\complement$. Using Corollary 1.2, we show here that the BNS invariant $\Sigma^1(X(\mathcal{A}), \mathbb{Z})$ is strictly contained in $S(\mathcal{R}^1(X(\mathcal{A}), \mathbb{R}))^\complement$. This answers a question first raised at an Oberwolfach Mini-Workshop [22], and revisited in [33, 36].

2. Characteristic and resonance varieties

We start with a brief review of the characteristic varieties of a space, and their relation to the resonance varieties, via two kinds of tangent cone constructions.

2.1. **Jump loci for twisted homology.** Let X be a connected CW-complex with finite k-skeleton, for some $k \geq 1$. Without loss of generality, we may assume X has a single 0-cell, call it x_0. Let $G = \pi_1(X, x_0)$ be the fundamental group of X, and let $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$ be the group of complex characters of G. Clearly, $\widehat{G} = \widehat{G}_{\mathrm{ab}}$, where $G_{\mathrm{ab}} = H_1(X, \mathbb{Z})$ is the abelianization of G. The universal coefficient theorem allows us to identify $\widehat{G} = H^1(X, \mathbb{C}^\times)$.

Each character $\rho\colon G \to \mathbb{C}^\times$ determines a rank 1 local system, $\mathbb{C}_\rho$, on our space X. Computing the homology groups of X with coefficients in such local systems carves out some interesting subsets of the character group.

Definition 2.1. The *characteristic varieties* of X are the sets

$$(5) \qquad \mathcal{V}_d^i(X) = \{\rho \in H^1(X, \mathbb{C}^\times) \mid \dim_\mathbb{C} H_i(X, \mathbb{C}_\rho) \geq d\}.$$

Clearly, $1 \in \mathcal{V}_d^i(X)$ if and only if $d \leq b_i(X)$. In degree 0, we have $\mathcal{V}_1^0(X) = \{1\}$ and $\mathcal{V}_d^0(X) = \emptyset$, for $d > 1$. In degree 1, the sets $\mathcal{V}_d^i(X)$ depend only on the group $G = \pi_1(X, x_0)$—in fact, only on its maximal metabelian quotient, G/G''.

For the purpose of computing the characteristic varieties up to degree $i = k$, we may assume without loss of generality that X is a finite CW-complex of dimension $k + 1$, see [33]. With that in mind, it can be shown that the jump loci $\mathcal{V}_d^i(X)$ are Zariski closed subsets of the algebraic group $H^1(X, \mathbb{C}^\times)$, and that they depend only on the homotopy type of X. For details and further references, see [38].

One may extend the definition of characteristic varieties to arbitrary fields $\Bbbk$. The resulting varieties, $\mathcal{V}_d^i(X, \Bbbk)$, behave well under field extensions: if $\Bbbk \subseteq \mathbb{K}$, then $\mathcal{V}_d^i(X, \Bbbk) = \mathcal{V}_d^i(X, \mathbb{K}) \cap H^1(X, \Bbbk^\times)$.

Most important for us are the depth one characteristic varieties, $\mathcal{V}_1^i(X)$, and their unions up to a fixed degree, $\mathcal{V}^i(X) = \bigcup_{j=0}^i \mathcal{V}_1^j(X)$. These varieties yield an ascending filtration of the character group,

$$(6) \qquad \{1\} = \mathcal{V}^0(X) \subseteq \mathcal{V}^1(X) \subseteq \cdots \subseteq \mathcal{V}^k(X) \subseteq \widehat{G}.$$

Now let $\widehat{G}^\circ$ be the identity component of the character group $\widehat{G}$. Writing $n = b_1(X)$, we may identify $\widehat{G}^\circ$ with the complex algebraic torus $(\mathbb{C}^\times)^n$. Set

$$(7) \qquad \mathcal{W}^i(X) = \mathcal{V}^i(X) \cap \widehat{G}^\circ.$$

These varieties yield an ascending filtration of the complex algebraic torus $\widehat{G}^\circ$,

$$(8) \qquad \{1\} = \mathcal{W}^0(X) \subseteq \mathcal{W}^1(X) \subseteq \cdots \subseteq \mathcal{W}^k(X) \subseteq (\mathbb{C}^\times)^n.$$

The characteristic varieties behave well with respect to direct products. For instance, suppose both X_1 and X_2 have finite k-skeleton. Then, by [33] (see also [38]),

we have that

$$\mathcal{W}^i(X_1 \times X_2) = \bigcup_{p+q=i} \mathcal{W}^p(X_1) \times \mathcal{W}^q(X_2), \tag{9}$$

for all $i \le k$.

2.2. **Tangent cones and exponential tangent cones.** Let $W \subset (\mathbb{C}^\times)^n$ be a Zariski closed subset, defined by an ideal I in the Laurent polynomial ring $\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. Picking a finite generating set for I, and multiplying these generators with suitable monomials if necessary, we see that W may also be defined by the ideal $I \cap R$ in the polynomial ring $R = \mathbb{C}[t_1, \ldots, t_n]$. Finally, let J be the ideal in the polynomial ring $S = \mathbb{C}[z_1, \ldots, z_n]$, generated by the polynomials $g(z_1, \ldots, z_n) = f(z_1 + 1, \ldots, z_n + 1)$, for all $f \in I \cap R$.

Definition 2.2. The *tangent cone* of W at 1 is the algebraic subset $\mathrm{TC}_1(W) \subset \mathbb{C}^n$ defined by the ideal $\mathrm{in}(J) \subset S$ generated by the initial forms of all non-zero elements from J.

The tangent cone $\mathrm{TC}_1(W)$ is a homogeneous subvariety of $\mathbb{C}^n$, which depends only on the analytic germ of W at the identity. In particular, $\mathrm{TC}_1(W) \ne \emptyset$ if and only if $1 \in W$. Moreover, TC_1 commutes with finite unions.

The following, related notion was introduced in [16], and further studied in [33] and [38].

Definition 2.3. The *exponential tangent cone* of W at 1 is the homogeneous subvariety $\tau_1(W)$ of $\mathbb{C}^n$, defined by

$$\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \text{ for all } \lambda \in \mathbb{C}\}. \tag{10}$$

Again, $\tau_1(W)$ depends only on the analytic germ of W at the identity, and so $\tau_1(W) \ne \emptyset$ if and only if $1 \in W$. Moreover, τ_1 commutes with finite unions, as well as arbitrary intersections. The most important properties of this construction are summarized in the following result from [16] (see also [33] and [38]).

Theorem 2.4. *For every Zariski closed subset $W \subset (\mathbb{C}^\times)^n$, the following hold:*
  (1) *$\tau_1(W)$ is a finite union of rationally defined linear subspaces of $\mathbb{C}^n$.*
  (2) *$\tau_1(W) \subseteq \mathrm{TC}_1(W)$.*

If W is an algebraic subtorus of $(\mathbb{C}^\times)^n$, then $\tau_1(W) = \mathrm{TC}_1(W)$, and both types of tangent cones coincide with the tangent space at the origin, $T_1(W)$; moreover, $W = \exp(\tau_1(W))$ in this case. More generally, if all positive-dimensional components of W are algebraic subtori, then $\tau_1(W) = \mathrm{TC}_1(W)$. In general, though, the inclusion from Theorem 2.4(2) can be strict.

For brevity, we shall write $\tau_1^{\mathbb{Q}}(W) = \mathbb{Q}^n \cap \tau_1(W)$ for the rational points on the exponential tangent cone, and $\tau_1^{\mathbb{R}}(W) = \mathbb{R}^n \cap \tau_1(W)$ for the real points.

The main example we have in mind is that of the characteristic varieties $\mathcal{W}^i(X)$, viewed as Zariski closed subsets of the algebraic torus $H^1(X, \mathbb{C}^\times)^\circ = (\mathbb{C}^\times)^n$, where $n = b_1(X)$. By Theorem 2.4, the exponential tangent cone to $\mathcal{W}^i(X)$ can be written as a union of rationally defined linear subspaces,

$$\tau_1(\mathcal{W}^i(X)) = \bigcup_{L \in \mathcal{C}_i(X)} L \otimes \mathbb{C}. \tag{11}$$

We call the resulting rational subspace arrangement, $\mathcal{C}_i(X)$, the *i-th characteristic arrangement* of X; evidently, $\tau_1^{\mathbb{Q}}(\mathcal{W}^i(X))$ is the union of this arrangement.

2.3. **Resonance varieties.** Now consider the cohomology algebra $A = H^*(X, \mathbb{C})$, with graded ranks the Betti numbers $b_i = \dim_{\mathbb{C}} A^i$. For each $a \in A^1$, we have $a^2 = 0$, by graded-commutativity of the cup product. Thus, right-multiplication by a defines a cochain complex,

$$(12) \qquad (A, \cdot a) \colon \ A^0 \xrightarrow{\ a\ } A^1 \xrightarrow{\ a\ } A^2 \longrightarrow \cdots,$$

The jump loci for the cohomology of this complex define a natural filtration of the affine space $A^1 = H^1(X, \mathbb{C})$.

Definition 2.5. The *resonance varieties* of X are the sets

$$(13) \qquad \mathcal{R}_d^i(X) = \{a \in A^1 \mid \dim_{\mathbb{C}} H^i(A, a) \geq d\}.$$

For the purpose of computing the resonance varieties in degrees $i \leq k$, we may assume without loss of generality that X is a finite CW-complex of dimension $k+1$. The sets $\mathcal{R}_d^i(X)$, then, are homogeneous, Zariski closed subsets of $A^1 = \mathbb{C}^n$, where $n = b_1$. In each degree $i \leq k$, the resonance varieties provide a descending filtration,

$$(14) \qquad H^1(X, \mathbb{C}) \supseteq \mathcal{R}_1^i(X) \supseteq \cdots \supseteq \mathcal{R}_{b_i}^i(X) \supseteq \mathcal{R}_{b_i+1}^i(X) = \emptyset.$$

Note that, if $A^i = 0$, then $\mathcal{R}_d^i(X) = \emptyset$, for all $d > 0$. In degree 0, we have $\mathcal{R}_1^0(X) = \{0\}$, and $\mathcal{R}_d^0(X) = \emptyset$, for $d > 1$. In degree 1, the varieties $\mathcal{R}_d^1(X)$ depend only on the fundamental group $G = \pi_1(X, x_0)$—in fact, only on the cup-product map $\cup \colon H^1(G, \mathbb{C}) \wedge H^1(G, \mathbb{C}) \to H^2(G, \mathbb{C})$.

One may extend the definition of resonance varieties to arbitrary fields $\Bbbk$, with the proviso that $H_1(X, \mathbb{Z})$ should be torsion-free, if $\Bbbk$ has characteristic 2. The resulting varieties, $\mathcal{R}_d^i(X, \Bbbk)$, behave well under field extensions: if $\Bbbk \subseteq \mathbb{K}$, then $\mathcal{R}_d^i(X, \Bbbk) = \mathcal{R}_d^i(X, \mathbb{K}) \cap H^1(X, \Bbbk)$. In particular, $\mathcal{R}_d^i(X, \mathbb{Q})$ is just the set of rational points on the integrally defined variety $\mathcal{R}_d^i(X) = \mathcal{R}_d^i(X, \mathbb{C})$.

Most important for us are the depth-1 resonance varieties, $\mathcal{R}_1^i(X)$, and their unions up to a fixed degree, $\mathcal{R}^i(X) = \bigcup_{j=0}^{i} \mathcal{R}_1^j(X)$. The latter varieties can be written as

$$(15) \qquad \mathcal{R}^i(X) = \{a \in A^1 \mid H^j(A, \cdot a) \neq 0, \text{ for some } j \leq i\}.$$

These algebraic sets provide an ascending filtration of the first cohomology group,

$$(16) \qquad \{0\} = \mathcal{R}^0(X) \subseteq \mathcal{R}^1(X) \subseteq \cdots \subseteq \mathcal{R}^k(X) \subseteq H^1(X, \mathbb{C}) = \mathbb{C}^n.$$

As noted in [33], the resonance varieties also behave well with respect to direct products: if both X_1 and X_2 have finite k-skeleton, then, for all $i \leq k$,

$$(17) \qquad \mathcal{R}^i(X_1 \times X_2) = \bigcup_{p+q=i} \mathcal{R}^p(X_1) \times \mathcal{R}^q(X_2).$$

2.4. **Tangent cone and resonance.** An important feature of the theory of cohomology jumping loci is the relationship between characteristic and resonance varieties, based on the tangent cone construction. A foundational result in this direction is the following theorem of Libgober [27], which generalizes an earlier result of Green and Lazarsfeld [24].

Theorem 2.6. *Let X be a connected CW-complex with finite k-skeleton. Then, for all $i \leq k$ and $d > 0$, the tangent cone at 1 to $\mathcal{W}_d^i(X)$ is included in $\mathcal{R}_d^i(X)$.*

Putting together Theorems 2.4(2) and 2.6, and using the fact that both types of tangent cone constructions commute with finite unions, we obtain an immediate corollary.

Corollary 2.7. *For each $i \leq k$, the following inclusions hold:*

(1) $\tau_1(\mathcal{W}_d^i(X)) \subseteq \mathrm{TC}_1(\mathcal{W}_d^i(X)) \subseteq \mathcal{R}_d^i(X)$, *for all $d > 0$.*

(2) $\tau_1(\mathcal{W}^i(X)) \subseteq \mathrm{TC}_1(\mathcal{W}^i(X)) \subseteq \mathcal{R}^i(X)$.

In general, the above inclusions may very well be strict. In the presence of formality, though, they become equalities, at least in degree $i = 1$.

2.5. **Formality.** Let us now collect some known facts on various formality notions. For further details and references, we refer to the recent survey [32].

Let X be a connected CW-complex with finite 1-skeleton. The space X is *formal* if there is a zig-zag of commutative, differential graded algebra quasi-isomorphisms connecting Sullivan's algebra of polynomial differential forms, $A_{\mathrm{PL}}(X, \mathbb{Q})$, to the rational cohomology algebra, $H^*(X, \mathbb{Q})$, endowed with the zero differential. The space X is merely *k-formal* (for some $k \geq 1$) if each of these morphisms induces an isomorphism in degrees up to k, and a monomorphism in degree $k + 1$.

Examples of formal spaces include rational cohomology tori, Riemann surfaces, compact connected Lie groups, as well as their classifying spaces. On the other hand, a nilmanifold is formal if and only if it is a torus. Formality is preserved under wedges and products of spaces, and connected sums of manifolds.

The 1-minimality property of a space depends only on its fundamental group. A finitely generated group G is said to be 1-formal if it admits a classifying space $K(G, 1)$ which is 1-formal, or, equivalently, if the Malcev Lie algebra $\mathfrak{m}(G)$ (that is, the Lie algebra of the rational, prounipotent completion of G) admits a quadratic presentation. Examples of 1-formal groups include free groups and free abelian groups of finite rank, surface groups, and groups with first Betti number equal to 0 or 1. The 1-formality property is preserved under free products and direct products.

Theorem 2.8 ([16]). *Let X be a 1-formal space. Then, for each $d > 0$,*

(18) $$\tau_1(\mathcal{V}_d^1(X)) = \mathrm{TC}_1(\mathcal{V}_d^1(X)) = \mathcal{R}_d^1(X).$$

In particular, the first resonance variety, $\mathcal{R}^1(X)$, of a 1-formal space X is a finite union of rationally defined linear subspaces.

2.6. **Straightness.** In [37], we delineate another class of spaces for which the resonance and characteristic varieties are intimately related to each other via the tangent cone constructions.

Definition 2.9. We say that X is *locally k-straight* if, for each $i \leq k$, all components of $\mathcal{W}^i(X)$ passing through the origin 1 are algebraic subtori, and the tangent cone at 1 to $\mathcal{W}^i(X)$ equals $\mathcal{R}^i(X)$. If, moreover, all positive-dimensional components of $\mathcal{W}^i(X)$ contain the origin, we say X is *k-straight*. If these conditions hold for all $k \geq 1$, we say X is *(locally) straight*.

Examples of straight spaces include Riemann surfaces, tori, and knot complements. Under some further assumptions, the straightness properties behave well with respect to finite direct products and wedges.

It follows from [16] that every 1-formal space is locally 1-straight. In general, though, examples from [37] show that 1-formal spaces need not be 1-straight, and 1-straight spaces need not be 1-formal.

Theorem 2.10 ([37]). *Let X be a locally k-straight space. Then, for all $i \leq k$,*

(1) $\tau_1(\mathcal{W}^i(X)) = \mathrm{TC}_1(\mathcal{W}^i(X)) = \mathcal{R}^i(X)$.

(2) $\mathcal{R}^i(X, \mathbb{Q}) = \bigcup_{L \in \mathcal{C}_i(X)} L$.

In particular, all the resonance varieties $\mathcal{R}^i(X)$ of a locally straight space X are finite unions of rationally defined linear subspaces.

3. The Dwyer–Fried invariants

In this section, we recall the definition of the Dwyer–Fried sets, and the way these sets relate to the (co)homology jump loci of a space.

3.1. **Betti numbers of free abelian covers.** As before, let X be a connected CW-complex with finite 1-skeleton, and let $G = \pi_1(X, x_0)$. Denote by $n = b_1(X)$ the first Betti number of X. Fix an integer r between 1 and n, and consider the (connected) regular covers of X, with group of deck-transformations $\mathbb{Z}^r$.

Each such cover, $X^\nu \to X$, is determined by an epimorphism $\nu \colon G \twoheadrightarrow \mathbb{Z}^r$. The induced homomorphism in rational cohomology, $\nu^* \colon H^1(\mathbb{Z}^r, \mathbb{Q}) \hookrightarrow H^1(G, \mathbb{Q})$, defines an r-dimensional subspace, $P_\nu = \mathrm{im}(\nu^*)$, in the n-dimensional $\mathbb{Q}$-vector space $H^1(G, \mathbb{Q}) = H^1(X, \mathbb{Q})$. Conversely, each r-dimensional subspace $P \subset H^1(X, \mathbb{Q})$ can be written as $P = P_\nu$, for some epimorphism $\nu \colon G \twoheadrightarrow \mathbb{Z}^r$, and thus defines a regular $\mathbb{Z}^r$-cover of X.

In summary, the regular $\mathbb{Z}^r$-covers of X are parametrized by the Grassmannian of r-planes in $H^1(X, \mathbb{Q})$, via the correspondence

$$(19) \qquad \begin{aligned} \{\text{regular } \mathbb{Z}^r\text{-covers of } X\} &\longleftrightarrow \{r\text{-planes in } H^1(X, \mathbb{Q})\} \\ X^\nu \to X \qquad &\longleftrightarrow \qquad P_\nu = \mathrm{im}(\nu^*). \end{aligned}$$

Moving about the rational Grassmannian and recording how the Betti numbers of the corresponding covers vary leads to the following definition.

Definition 3.1. The *Dwyer–Fried invariants* of the space X are the subsets

$$\Omega_r^i(X) = \left\{ P_\nu \in \mathrm{Gr}_r(H^1(X, \mathbb{Q})) \mid b_j(X^\nu) < \infty \text{ for } j \leq i \right\}.$$

For a fixed integer r between 1 and n, these sets form a descending filtration of the Grassmannian of r-planes in $H^1(X, \mathbb{Q}) = \mathbb{Q}^n$,

$$(20) \qquad \mathrm{Gr}_r(\mathbb{Q}^n) = \Omega_r^0(X) \supseteq \Omega_r^1(X) \supseteq \Omega_r^2(X) \supseteq \cdots .$$

If $r > n$, we adopt the convention that $\mathrm{Gr}_r(\mathbb{Q}^n) = \emptyset$, and define $\Omega_r^i(X) = \emptyset$.

As noted in [38], the Ω-sets are invariants of homotopy-type: if $f \colon X \to Y$ is a homotopy equivalence, then the induced isomorphism in cohomology, $f^* \colon H^1(Y, \mathbb{Q}) \to H^1(X, \mathbb{Q})$, defines isomorphisms $f_r^* \colon \mathrm{Gr}_r(H^1(Y, \mathbb{Q})) \to \mathrm{Gr}_r(H^1(X, \mathbb{Q}))$, which send each subset $\Omega_r^i(Y)$ bijectively onto $\Omega_r^i(X)$.

Particularly simple is the situation when $n = b_1(X) > 0$ and $r = n$. In this case, $\mathrm{Gr}_n(H^1(X, \mathbb{Q})) = \{\mathrm{pt}\}$. Under the correspondence from (19), this single point is realized by the maximal free abelian cover, $X^\alpha \to X$, where $\alpha \colon G \twoheadrightarrow G_{\mathrm{ab}} / \mathrm{Tors}(G_{\mathrm{ab}}) =$

$\mathbb{Z}^n$ is the canonical projection. The sets $\Omega_n^i(X)$ are then given by

$$(21) \qquad \Omega_n^i(X) = \begin{cases} \{\text{pt}\} & \text{if } b_j(X^\alpha) < \infty \text{ for all } j \leq i, \\ \emptyset & \text{otherwise.} \end{cases}$$

3.2. **Dwyer–Fried invariants and characteristic varieties.** The next theorem reduces the computation of the Ω-sets to a more standard computation in algebraic geometry. The theorem was proved by Dwyer and Fried in [17], using the support loci for the Alexander invariants, and was recast in a slightly more general context by Papadima and Suciu in [33], using the characteristic varieties. We state this result in the form established in [38].

Theorem 3.2 ([17, 33, 38]). *Suppose X has finite k-skeleton, for some $k \geq 1$. Then, for all $i \leq k$ and $r \geq 1$,*

$$(22) \qquad \Omega_r^i(X) = \big\{ P \in \mathrm{Gr}_r(\mathbb{Q}^n) \mid \dim\big(\exp(P \otimes \mathbb{C}) \cap \mathcal{W}^i(X)\big) = 0 \big\}.$$

In other words, an r-plane $P \subset \mathbb{Q}^n$ belongs to the set $\Omega_r^i(X)$ if and only if the algebraic torus $T = \exp(P \otimes \mathbb{C})$ intersects the characteristic variety $W = \mathcal{W}^i(X)$ only in finitely many points. When this happens, the exponential tangent cone $\tau_1(T \cap W)$ equals $\{0\}$, forcing $P \cap L = \{0\}$, for every subspace $L \subset \mathbb{Q}^n$ in the characteristic subspace arrangement $\mathcal{C}_i(X)$. Consequently,

$$(23) \qquad \Omega_r^i(X) \subseteq \bigg(\bigcup_{L \in \mathcal{C}_i(X)} \big\{ P \in \mathrm{Gr}_r(H^1(X, \mathbb{Q})) \mid P \cap L \neq \{0\} \big\} \bigg)^{\complement}.$$

As noted in [38], this inclusion may be reinterpreted in terms of the classical incidence correspondence from algebraic geometry. Let V be a homogeneous variety in $\Bbbk^n$. The locus of r-planes in $\Bbbk^n$ that meet V,

$$(24) \qquad \sigma_r(V) = \big\{ P \in \mathrm{Gr}_r(\Bbbk^n) \mid P \cap V \neq \{0\} \big\},$$

is a Zariski closed subset of the Grassmannian $\mathrm{Gr}_r(\Bbbk^n)$. In the case when V is a linear subspace $L \subset \Bbbk^n$, the incidence variety $\sigma_r(L)$ is known as the *special Schubert variety* defined by L. If L has codimension d in $\Bbbk^n$, then $\sigma_r(L)$ has codimension $d - r + 1$ in $\mathrm{Gr}_r(\Bbbk^n)$.

Applying this discussion to the homogeneous variety $\tau_1^{\mathbb{Q}}(\mathcal{W}^i(X)) = \bigcup_{L \in \mathcal{C}_i(X)} L$ lying inside $H^1(X, \mathbb{Q}) = \mathbb{Q}^n$, and using formula (23), we obtain the following corollary.

Corollary 3.3 ([38]). *Let X be a connected CW-complex with finite k-skeleton. For all $i \leq k$ and $r \geq 1$,*

$$(25) \qquad \begin{aligned} \Omega_r^i(X) &\subseteq \mathrm{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r\big(\tau_1^{\mathbb{Q}}(\mathcal{W}^i(X))\big) \\ &= \mathrm{Gr}_r(H^1(X, \mathbb{Q})) \setminus \bigcup_{L \in \mathcal{C}_i(X)} \sigma_r(L). \end{aligned}$$

In other words, each set $\Omega_r^i(X)$ is contained in the complement of a Zariski closed subset of the Grassmanian $\mathrm{Gr}_r(H^1(X, \mathbb{Q}))$, namely, the union of the special Schubert varieties $\sigma_r(L)$ corresponding to the subspaces L in $\mathcal{C}_i(X)$.

Under appropriate hypothesis, the inclusion from Corollary 3.3 holds as equality. The next two propositions illustrate this point.

Proposition 3.4 ([38]). *Let X be a connected CW-complex with finite k-skeleton. Suppose that, for some $i \leq k$, all positive-dimensional components of $\mathcal{W}^i(X)$ are algebraic subtori. Then, for all $r \geq 1$,*

$$(26) \qquad \Omega_r^i(X) = \mathrm{Gr}_r(H^1(X, \mathbb{Q})) \setminus \bigcup_{L \in \mathcal{C}_i(X)} \sigma_r(L).$$

Proposition 3.5 ([17], [33], [38]). *Let X be a CW-complex with finite k-skeleton. Then, for all $i \leq k$,*

$$(27) \qquad \Omega_1^i(X) = \mathbb{P}(H^1(X, \mathbb{Q})) \setminus \bigcup_{L \in \mathcal{C}_i(X)} \mathbb{P}(L),$$

where $\mathbb{P}(V)$ denotes the projectivization of a homogeneous subvariety $V \subseteq H^1(X, \mathbb{Q})$.

In either of these two situations, the sets $\Omega_r^i(X)$ are Zariski open subsets of $\mathrm{Gr}_r(H^1(X, \mathbb{Q}))$. In general, though, the sets $\Omega_r^i(X)$ need not be open, not even in the usual topology on the rational Grassmanian. This phenomenon was first noticed by Dwyer and Fried, who constructed in [17] a 3-dimensional cell complex X for which $\Omega_2^2(X)$ is a finite set (see Example 5.12). In [38], we provide examples of finitely presented groups G for which $\Omega_2^1(G)$ is not open.

3.3. Straightness and the Ω-sets.
Under appropriate straightness assumptions, the upper bounds for the Dwyer–Fried sets can be expressed in terms of the resonance varieties associated to the cohomology ring $H^*(X, \mathbb{Q})$.

Theorem 3.6 ([37]). *Let X be a connected CW-complex.*
  (1) *If X is locally k-straight, then $\Omega_r^i(X) \subseteq \mathrm{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r(\mathcal{R}^i(X, \mathbb{Q}))$, for all $i \leq k$ and $r \geq 1$.*
  (2) *If X is k-straight, then $\Omega_r^i(X) = \mathrm{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r(\mathcal{R}^i(X, \mathbb{Q}))$, for all $i \leq k$ and $r \geq 1$.*

If X is locally k-straight, we also know from Theorem 2.10(2) that $\mathcal{R}^i(X, \mathbb{Q})$ is the union of the linear subspaces comprising $\mathcal{C}_i(X)$, for all $i \leq k$. Thus, if X is k-straight, then $\Omega_r^i(X)$ is the complement of a finite union of special Schubert varieties in the Grassmannian of r-planes in $H^1(X, \mathbb{Q})$; in particular, $\Omega_r^i(X)$ is a Zariski open set in $\mathrm{Gr}_r(H^1(X, \mathbb{Q}))$.

The straightness hypothesis is crucial for the equality in Theorem 3.6(2) to hold.

Example 3.7. Let $G = \langle x_1, x_2 \mid x_1^2 x_2 = x_2 x_1^2 \rangle$. Then $\mathcal{W}^1(G) = \{1\} \cup \{t \in (\mathbb{C}^\times)^2 \mid t_1 = -1\}$, while $\mathcal{R}^1(G) = \{0\}$. Thus, G is locally 1-straight, but not 1-straight. Moreover, $\Omega_2^1(G) = \emptyset$, yet $\sigma_2(\mathcal{R}^1(G, \mathbb{Q}))^{\complement} = \{\mathrm{pt}\}$.

4. The Bieri–Neumann–Strebel–Renz invariants

We now go over the several definitions of the Σ-invariants of a space X (and, in particular, of a group G), and discuss the way these invariants relate to the (co)homology jumping loci.

4.1. A finite type property.
We start with a finiteness condition for chain complexes, following the approach of Farber, Geoghegan, and Schütz [21]. Let $C = (C_i, \partial_i)_{i \geq 0}$ be a non-negatively graded chain complex over a ring R, and let k be a non-negative integer.

Definition 4.1. We say C is of *finite k-type* if there is a chain complex C' of finitely generated, projective R-modules and a k-equivalence between C' and C, i.e., a chain map $C' \to C$ inducing isomorphisms $H_i(C') \to H_i(C)$ for $i < k$ and an epimorphism $H_k(C') \to H_k(C)$.

Equivalently, there is a chain complex of free R-modules, call it D, such that D_i is finitely generated for all $i \leq k$, and there is a chain map $D \to C$ inducing an isomorphism $H_*(D) \to H_*(C)$. When C itself is free, we have the following alternate characterization from [21].

Lemma 4.2. *Let C be a free chain complex over a ring R. Then C is of finite k-type if and only if C is chain-homotopy equivalent to a chain complex D of free R-modules, such that D_i is finitely generated for all $i \leq k$.*

Remark 4.3. Suppose $\rho \colon R \to S$ is a ring morphism, and C is a chain complex over R. Then $C \otimes_R S$ naturally acquires the structure of an S-chain complex via extension of scalars. Now, if C is free, and of finite k-type over R, it is readily seen that $C \otimes_R S$ is of finite k-type over S.

4.2. **The Σ-invariants of a chain complex.** Let G be a finitely generated group, and denote by $\operatorname{Hom}(G, \mathbb{R})$ the set of homomorphisms from G to the additive group of the reals. Clearly, this is a finite-dimensional $\mathbb{R}$-vector space, on which the multiplicative group of positive reals naturally acts. After fixing an inner product on $\operatorname{Hom}(G, \mathbb{R})$, the quotient space,

$$(28) \qquad\qquad S(G) = (\operatorname{Hom}(G, \mathbb{R}) \setminus \{0\})/\mathbb{R}^+,$$

may be identified with the unit sphere in $\operatorname{Hom}(G, \mathbb{R})$. Up to homeomorphism, this sphere is determined by the first Betti number of G. Indeed, if $b_1(G) = n$, then $S(G) = S^{n-1}$; in particular, if $b_1(G) = 0$, then $S(G) = \emptyset$. To simplify notation, we will denote both a non-zero homomorphism $G \to \mathbb{R}$ and its equivalence class in $S(G)$ by the same symbol, and we will routinely view $S(G)$ as embedded in $\operatorname{Hom}(G, \mathbb{R})$.

Given a homomorphism $\chi \colon G \to \mathbb{R}$, consider the set $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$. Clearly, G_χ is a submonoid of G, and the monoid ring $\mathbb{Z}G_\chi$ is a subring of the group ring $\mathbb{Z}G$. Thus, any $\mathbb{Z}G$-module naturally acquires the structure of a $\mathbb{Z}G_\chi$-module, by restriction of scalars.

Definition 4.4 ([21])**.** Let C be a chain complex over $\mathbb{Z}G$. For each integer $k \geq 0$, the *k-th Bieri–Neumann–Bieri–Strebel invariant* of C is the set

$$(29) \qquad\qquad \Sigma^k(C) = \{\chi \in S(G) \mid C \text{ is of finite } k\text{-type over } \mathbb{Z}G_\chi\}.$$

Note that $\mathbb{Z}G$ is a flat $\mathbb{Z}G_\chi$-module, and $\mathbb{Z}G \otimes_{\mathbb{Z}G_\chi} M \cong M$ for every $\mathbb{Z}G$-module M. Thus, if $\Sigma^k(C)$ is non-empty, then C must be of finite k-type over $\mathbb{Z}G$.

To a large extent, the importance of the Σ-invariants lies in the fact that they control the finiteness properties of kernels of projections to abelian quotients. More precisely, let N be a normal subgroup of G, with G/N abelian. Define

$$(30) \qquad\qquad S(G, N) = \{\chi \in S(G) \mid N \leq \ker(\chi)\}.$$

It is readily seen that $S(G, N)$ is the great subsphere obtained by intersecting the unit sphere $S(G) \subset H^1(G, \mathbb{R})$ with the linear subspace $P_\nu \otimes \mathbb{R}$, where $\nu \colon G \twoheadrightarrow G/N$ is the canonical projection, and $P_\nu = \operatorname{im}(\nu^* \colon H^1(G/N, \mathbb{Q}) \hookrightarrow H^1(G, \mathbb{Q}))$. Notice also that every $\mathbb{Z}G$-module acquires the structure of a $\mathbb{Z}N$-module by restricting scalars.

Theorem 4.5 ([21])**.** *Let C be a chain complex of free $\mathbb{Z}G$-modules, with C_i finitely generated for $i \leq k$, and let N be a normal subgroup of G, with G/N is abelian. Then C is of finite k-type over $\mathbb{Z}N$ if and only if $S(G, N) \subset \Sigma^k(C)$.*

4.3. **The Σ-invariants of a CW-complex.** Let X be a connected CW-complex with finite 1-skeleton, and let $G = \pi_1(X, x_0)$ be its fundamental group. Picking a classifying map $X \to K(G, 1)$, we obtain an induced isomorphism, $H^1(G, \mathbb{R}) \xrightarrow{\cong} H^1(X, \mathbb{R})$, which identifies the respective unit spheres, $S(G)$ and $S(X)$.

The cell structure on X lifts to a cell structure on the universal cover $\widetilde{X}$. Clearly, this lifted cell structure is invariant under the action of G by deck transformations. Thus, the cellular chain complex $C_*(\widetilde{X}, \mathbb{Z})$ is a chain complex of (free) $\mathbb{Z}G$-modules.

Definition 4.6. For each $k \geq 0$, the *k-th Bieri–Neumann–Strebel–Renz invariant* of X is the subset of $S(X)$ given by

$$(31) \qquad \Sigma^k(X, \mathbb{Z}) = \Sigma^k(C_*(\widetilde{X}, \mathbb{Z})).$$

It is shown in [21] that $\Sigma^k(X, \mathbb{Z})$ is an open subset of $S(X)$, which depends only on the homotopy type of the space X. Clearly, $\Sigma^0(X, \mathbb{Z}) = S(X)$.

Now let G be a finitely generated group, and pick a classifying space $K(G, 1)$. In view of the above discussion, the sets

$$(32) \qquad \Sigma^k(G, \mathbb{Z}) := \Sigma^k(K(G, 1), \mathbb{Z}),$$

are well-defined invariants of the group G. These sets, which live inside the unit sphere $S(G) \subset \mathrm{Hom}(G, \mathbb{R})$, coincide with the classical geometric invariants of Bieri, Neumann and Strebel [7] and Bieri and Renz [8].

The Σ-invariants of a CW-complex and those of its fundamental group are related, as follows.

Proposition 4.7 ([21])**.** *Let X be a connected CW-complex with finite 1-skeleton. If $\widetilde{X}$ is k-connected, then*

$$(33) \qquad \Sigma^k(X) = \Sigma^k(\pi_1(X)), \;\; and \;\; \Sigma^{k+1}(X) \subseteq \Sigma^{k+1}(\pi_1(X)).$$

The last inclusion can of course be strict. For instance, if $X = S^1 \vee S^{k+1}$, with $k \geq 1$, then $\Sigma^{k+1}(X) = \emptyset$, though $\Sigma^{k+1}(\pi_1(X)) = S^0$. We shall see a more subtle occurrence of this phenomenon in Example 5.12.

4.4. **Generalizations and discussion.** More generally, if M is a $\mathbb{Z}G$-module, the invariants $\Sigma^k(G, M)$ of Bieri and Renz [8] are given by $\Sigma^k(G, M) = \Sigma^i(F_\bullet)$, where $F_\bullet \to M$ is a projective $\mathbb{Z}G$-resolution of M. In particular, we have the invariants $\Sigma^k(G, \Bbbk)$, where $\Bbbk$ is a field, viewed as a trivial $\mathbb{Z}G$-module. There is always an inclusion $\Sigma^k(G, \mathbb{Z}) \subseteq \Sigma^i(G, \Bbbk)$, but this inclusion may be strict, as we shall see in Example 6.5 below.

An alternate definition of the Σ-invariants of a group is as follows. Recall that a monoid (in particular, a group) G is of type FP_k if there is a projective $\mathbb{Z}G$-resolution $F_\bullet \to \mathbb{Z}$, with F_i finitely generated, for all $i \leq k$. In particular, G is of type FP_1 if and only if G is finitely generated. We then have

$$(34) \qquad \Sigma^k(G, \mathbb{Z}) = \{\chi \in S(G) \mid \text{the monoid } G_\chi \text{ is of type } \mathrm{FP}_k\}.$$

These sets form a descending chain of open subsets of $S(G)$, starting at $\Sigma^1(G) = \Sigma^1(G, \mathbb{Z})$. Moreover, $\Sigma^k(G, \mathbb{Z})$ is non-empty only if G is of type FP_k.

If N is a normal subgroup of G, with G/N abelian, then Theorem 4.5 implies the following result from [7, 8]: The group N is of type FP_k if and only if $S(G,N) \subseteq \Sigma^k(G,\mathbb{Z})$. In particular, the kernel of an epimorphism $\chi\colon G \twoheadrightarrow \mathbb{Z}$ is finitely generated if and only if both χ and $-\chi$ belong to $\Sigma^1(G)$.

One class of groups for which the BNSR invariants can be computed explicitly is that of one-relator groups. K. Brown gave in [9] an algorithm for computing $\Sigma^1(G)$ for groups G in this class, while Bieri and Renz in [8] reinterpreted this algorithm in terms of Fox calculus, and showed that, for 1-relator groups, $\Sigma^k(G,\mathbb{Z}) = \Sigma^1(G)$, for all $k \geq 2$.

Another class of groups for which the Σ-invariants can be completely determined is that of non-trivial free products. Indeed, if G_1 and G_2 are two non-trivial, finitely generated groups, then $\Sigma^k(G_1 * G_2, \mathbb{Z}) = \emptyset$, for all $k \geq 1$ (see, for instance, [33]).

Finally, it should be noted that the BNSR invariants obey some very nice product formulas. For instance, let G_1 and G_2 be two groups of type F_k, for some $k \geq 1$. Identify the sphere $S(G_1 \times G_2)$ with the join $S(G_1) * S(G_2)$. Then, for all $i \leq k$,

$$(35) \qquad \Sigma^i(G_1 \times G_2, \mathbb{Z})^\complement \subseteq \bigcup_{p+q=i} \Sigma^p(G_1, \mathbb{Z})^\complement * \Sigma^q(G_2, \mathbb{Z})^\complement,$$

where again $A * B$ denotes the join of two spaces, with the convention that $A * \emptyset = A$. As shown in [7, Theorem 7.4], the above inclusion holds as equality for $i = 1$, i.e.,

$$(36) \qquad \Sigma^1(G_1 \times G_2, \mathbb{Z})^\complement = \Sigma^1(G_1, \mathbb{Z})^\complement \cup \Sigma^1(G_2, \mathbb{Z})^\complement.$$

The general formula (35) was established by Meinert (unpublished) and Gehrke [23]. Recently, it was shown by Schütz [34] and Bieri–Geoghegan [6] that equality holds in (35) for all $i \leq 3$, although equality may fail for $i \geq 4$. Furthermore, it was shown in [6] that the analogous product formula for the Σ-invariants with coefficients in a field $\Bbbk$ holds as an equality, for all $i \leq k$.

4.5. **Novikov homology.** In his 1987 thesis, J.-Cl. Sikorav reinterpreted the BNS invariant of a finitely generated group in terms of Novikov homology. This interpretation was extended to all BNSR invariants by Bieri [5], and later to the BNSR invariants of CW-complexes by Farber, Geoghegan and Schütz [21].

The *Novikov–Sikorav completion* of the group ring $\mathbb{Z}G$ with respect to a homomorphism $\chi\colon G \to \mathbb{R}$ consists of all formal sums $\sum_j n_j g_j$, with $n_j \in \mathbb{Z}$ and $g_j \in G$, having the property that, for each $c \in \mathbb{R}$, the set of indices j for which $n_j \neq 0$ and $\chi(g_j) \geq c$ is finite. With the obvious addition and multiplication, the Novikov–Sikorav completion, $\widehat{\mathbb{Z}G}_\chi$, is a ring, containing $\mathbb{Z}G$ as a subring; in particular, $\widehat{\mathbb{Z}G}_\chi$ carries a natural G-module structure. We refer to M. Farber's book [20] for a comprehensive treatment, and to R. Bieri [5] for further details.

Let X be a connected CW-complex with finite k-skeleton, and let $G = \pi_1(X, x_0)$ be its fundamental group.

Theorem 4.8 ([21])**.** *With notation as above,*

$$(37) \qquad \Sigma^k(X, \mathbb{Z}) = \{\chi \in S(X) \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0, \text{ for all } i \leq k\}.$$

Now, every non-zero homomorphism $\chi\colon G \to \mathbb{R}$ factors as $\chi = \iota \circ \xi$, where $\xi\colon G \twoheadrightarrow \Gamma$ is a surjection onto a lattice $\Gamma \cong \mathbb{Z}^r$ in $\mathbb{R}$, and $\iota\colon \Gamma \hookrightarrow \mathbb{R}$ is the inclusion map. A Laurent polynomial $p = \sum_\gamma n_\gamma \gamma \in \mathbb{Z}\Gamma$ is said to be ι-monic if the greatest element in $\iota(\mathrm{supp}(p))$ is 0, and $n_0 = 1$; every such polynomial is invertible in the completion

$\widehat{\mathbb{Z}\Gamma}_\iota$. We denote by $\mathcal{R}\Gamma_\iota$ the localization of $\mathbb{Z}\Gamma$ at the multiplicative subset of all ι-monic polynomials. Using the known fact that $\mathcal{R}\Gamma_\iota$ is both a G-module and a PID, one may define for each $i \leq k$ the i-th *Novikov Betti number*, $b_i(X, \chi)$, as the rank of the finitely generated $\mathcal{R}\Gamma_\iota$-module $H_i(X, \mathcal{R}\Gamma_\iota)$.

4.6. **Σ-invariants and characteristic varieties.** The following result from [33] creates a bridge between the Σ-invariants of a space and the real points on the exponential tangent cones to the respective characteristic varieties.

Theorem 4.9 ([33])**.** *Let X be a CW-complex with finite k-skeleton, for some $k \geq 1$, and let $\chi \in S(X)$. The following then hold.*

    (1) *If $-\chi$ belongs to $\Sigma^k(X, \mathbb{Z})$, then $H_i(X, \mathcal{R}\Gamma_\iota) = 0$, and so $b_i(X, \chi) = 0$, for all $i \leq k$.*

    (2) *χ does not belong to $\tau_1^{\mathbb{R}}(\mathcal{W}^k(X))$ if and only if $b_i(X, \chi) = 0$, for all $i \leq k$.*

Recall that $S(V)$ denotes the intersection of the unit sphere $S(X)$ with a homogeneous subvariety $V \subset H^1(X, \mathbb{R})$. In particular, if $V = \{0\}$, then $S(V) = \emptyset$.

Corollary 4.10 ([33])**.** *Let X be a CW-complex with finite k-skeleton. Then, for all $i \leq k$,*

$$\text{(38)} \qquad \Sigma^i(X, \mathbb{Z}) \subseteq S(X) \setminus S(\tau_1^{\mathbb{R}}(\mathcal{W}^i(X))).$$

Qualitatively, Corollary 4.10 says that each BNSR set $\Sigma^i(X, \mathbb{Z})$ is contained in the complement of a union of rationally defined great subspheres.

As noted in [33], the above bound is sharp. For example, if X is a nilmanifold, then $\Sigma^i(X, \mathbb{Z}) = S(X)$, while $\mathcal{W}^i(X, \mathbb{C}) = \{1\}$, and so $\tau_1^{\mathbb{R}}(\mathcal{W}^i(X)) = \{0\}$, for all $i \geq 1$. Thus, the inclusion from Corollary 4.10 holds as an equality in this case.

If the space X is (locally) straight, Theorem 2.10 allows us to replace the exponential tangent cone by the corresponding resonance variety.

Corollary 4.11. *If X is locally k-straight, then, for all $i \leq k$,*

$$\text{(39)} \qquad \Sigma^i(X, \mathbb{Z}) \subseteq S(X) \setminus S(\mathcal{R}^i(X, \mathbb{R})).$$

Corollary 4.12 ([33])**.** *If G is a 1-formal group, then*

$$\text{(40)} \qquad \Sigma^1(G) \subseteq S(G) \setminus S(\mathcal{R}^1(G, \mathbb{R})).$$

5. RELATING THE Ω-INVARIANTS AND THE Σ-INVARIANTS

In this section, we prove Theorem 1.1 from the Introduction, which essentially says the following: if inclusion (25) is strict, then inclusion (38) is also strict.

5.1. **Finiteness properties of abelian covers.** Let X be a connected CW-complex with finite 1-skeleton, and let $\widetilde{X}$ be the universal cover, with group of deck transformation $G = \pi_1(X, x_0)$.

Let $p \colon X^\nu \to X$ be a (connected) regular, free abelian cover, associated to an epimorphism $\nu \colon G \twoheadrightarrow \mathbb{Z}^r$. Fix a basepoint $\tilde{x}_0 \in p^{-1}(x_0)$, and identify the fundamental group $\pi_1(X^\nu, \tilde{x}_0)$ with $N = \ker(\nu)$. Note that the universal cover $\widetilde{X^\nu}$ is homeomorphic to $\widetilde{X}$.

Finally, set

$$\text{(41)} \qquad S(X, X^\nu) = \{\chi \in S(X) \mid p^*(\chi) = 0\}.$$

Then $S(X, X^\nu)$ is a great sphere of dimension $r - 1$. In fact,

$$(42) \qquad\qquad S(X, X^\nu) = S(X) \cap (P_\nu \otimes \mathbb{R}),$$

where recall P_ν is the r-plane in $H^1(X, \mathbb{Q})$ determined by ν via the correspondence from (19). Moreover, under the identification $S(X) = S(G)$, the subsphere $S(X, X^\nu)$ corresponds to $S(G, N)$.

Proposition 5.1. *Let X be a connected CW-complex with finite k-skeleton, and let $p \colon X^\nu \to X$ be a regular, free abelian cover. Then the chain complex $C_*(\widetilde{X^\nu}, \mathbb{Z})$ is of finite k-type over $\mathbb{Z}N$ if and only if $S(X, X^\nu) \subseteq \Sigma^k(X, \mathbb{Z})$.*

Proof. Consider the free $\mathbb{Z}G$-chain complex $C = C_*(\widetilde{X}, \mathbb{Z})$. Upon restricting scalars to the subring $\mathbb{Z}N \subset \mathbb{Z}G$, the resulting $\mathbb{Z}N$-chain complex may be identified with $C_*(\widetilde{X^\nu}, \mathbb{Z})$. The desired conclusion follows from Theorem 4.5. $\qquad\square$

5.2. **Upper bounds for the Σ- and Ω-invariants.** Now assume X has finite k-skeleton, for some $k \geq 1$. As we just saw, great subspheres in the Σ-invariants indicate directions in which the corresponding covers have good finiteness properties. The next result compares the rational points on such subspheres with the corresponding Ω-invariants.

Proposition 5.2. *Let $P \subseteq H^1(X, \mathbb{Q})$ be an r-dimensional linear subspace. If the unit sphere in P is included in $\Sigma^k(X, \mathbb{Z})$, then P belongs to $\Omega_r^k(X)$.*

Proof. Realize $P = P_\nu$, for some epimorphism $\nu \colon G \twoheadrightarrow \mathbb{Z}^r$, and let $X^\nu \to X$ be the corresponding $\mathbb{Z}^r$-cover. By assumption, $S(X, X^\nu) \subseteq \Sigma^k(X, \mathbb{Z})$. Thus, by Proposition 5.1, the chain complex $C_*(\widetilde{X^\nu}, \mathbb{Z})$ is of finite k-type over $\mathbb{Z}N$.

Now, in view of Remark 4.3, the chain complex $C_*(X^\nu, \mathbb{Z}) = C_*(\widetilde{X^\nu}, \mathbb{Z}) \otimes_{\mathbb{Z}N} \mathbb{Z}$ is of finite k-type over $\mathbb{Z}$. Therefore, $b_i(X^\nu) < \infty$, for all $i \leq k$. Hence, $P \in \Omega_r^k(X)$. $\quad\square$

Corollary 5.3. *Let $P \subseteq H^1(X, \mathbb{Q})$ be a linear subspace. If $S(P) \subseteq \Sigma^k(X, \mathbb{Z})$, then $P \cap \tau_1^{\mathbb{Q}}(\mathcal{W}^k(X)) = \{0\}$.*

Proof. Set $r = \dim_{\mathbb{Q}} P$. From Corollary 3.3, we know that $\Omega_r^k(X)$ is contained in the complement in $\mathrm{Gr}_r(H^1(X, \mathbb{Q}))$ to the incidence variety $\sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{W}^k(X)))$. The conclusion follows from Proposition 5.2. $\qquad\square$

We are now in a position to state and prove the main result of this section, which is simply a restatement of Theorem 1.1 from the Introduction.

Theorem 5.4. *If X is a connected CW-complex with finite k-skeleton, then, for all $r \geq 1$,*

$$(43) \qquad \Sigma^k(X, \mathbb{Z}) = S(\tau_1^{\mathbb{R}}(\mathcal{W}^k(X)))^{\complement} \implies \Omega_r^k(X) = \sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{W}^k(X)))^{\complement}.$$

Proof. From Corollary 3.3, we know that $\Omega_r^k(X) \subseteq \sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{W}^k(X)))^{\complement}$. Suppose this inclusion is strict. There is then an r-dimensional linear subspace $P \subseteq H^1(X, \mathbb{Q})$ such that

    (1) $P \cap \tau_1^{\mathbb{Q}}(\mathcal{W}^k(X)) = \{0\}$, and
    (2) $P \notin \Omega_r^k(X)$.

By supposition (2) and Proposition 5.2, we must have $S(P) \not\subseteq \Sigma^k(X, \mathbb{Z})$. Thus, there exists an element $\chi \in S(P)$ such that $\chi \notin \Sigma^k(X, \mathbb{Z})$.

Now, supposition (1) and the fact that $\chi \in P$ imply that $\chi \notin \tau_1^{\mathbb{Q}}(\mathcal{W}^k(X))$. Since χ belongs to $H^1(X, \mathbb{Q})$, we infer that $\chi \notin \tau_1^{\mathbb{R}}(\mathcal{W}^k(X))$.

We have shown that $\Sigma^k(X, \mathbb{Z}) \subsetneq S(\tau_1^{\mathbb{R}}(\mathcal{W}^k(X)))^\complement$, a contradiction. $\qquad\square$

Recall now that the incidence variety $\sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{W}^k(X)))$ is a Zariski closed subset of the Grassmannian $\mathrm{Gr}_r(H^1(X, \mathbb{Q}))$. Recall also that the Dwyer–Fried sets $\Omega_1^k(X)$ are Zariski open, but that $\Omega_r^k(X)$ is not necessarily open, if $1 < r < b_1(X)$. We thus have the following immediate corollary.

Corollary 5.5. *Suppose there is an integer $r \geq 2$ such that $\Omega_r^k(X)$ is not Zariski open in $\mathrm{Gr}_r(H^1(X, \mathbb{Q}))$. Then $\Sigma^k(X, \mathbb{Z}) \neq S(\tau_1^{\mathbb{R}}(\mathcal{W}^k(X)))^\complement$.*

5.3. **The straight and formal settings.** When the space X is locally k-straight, we may replace in the above the exponential tangent cone to the k-th characteristic variety of X by the corresponding resonance variety.

Corollary 5.6. *Let X be a locally k-straight space. Then, for all $r \geq 1$,*
$$\Sigma^k(X, \mathbb{Z}) = S(X) \setminus S(\mathcal{R}^k(X, \mathbb{R})) \implies \Omega_r^k(X) = \mathrm{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r(\mathcal{R}^k(X, \mathbb{Q})).$$

Proof. Follows at once from Theorems 5.4 and 2.10. $\qquad\square$

Recalling now that every 1-formal space is locally 1-straight (cf. Theorem 2.8), we derive the following corollary.

Corollary 5.7. *Let X be a 1-formal space. Then, for all $r \geq 1$,*
$$\Sigma^1(X, \mathbb{Z}) = S(X) \setminus S(\mathcal{R}^1(X, \mathbb{R})) \implies \Omega_r^1(X) = \mathrm{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r(\mathcal{R}^1(X, \mathbb{Q})).$$

Using Corollary 5.3 and Theorem 2.8, we obtain the following consequences, which partially recover Corollary 4.12.

Corollary 5.8. *Let X be a 1-formal space, and $P \subseteq H^1(X, \mathbb{Q})$ a linear subspace. If the unit sphere in P is included in $\Sigma^1(X)$, then $P \cap \mathcal{R}^1(X, \mathbb{Q}) = \{0\}$.*

Corollary 5.9. *Let G be a 1-formal group, and let $\chi \colon G \to \mathbb{Z}$ be a non-zero homomorphism. If $\{\pm\chi\} \subseteq \Sigma^1(G)$, then $\chi \notin \mathcal{R}^1(X, \mathbb{Q})$.*

The formality assumption is really necessary in the previous two corollaries. For instance, let X be the Heisenberg nilmanifold, i.e., the S^1-bundle over $S^1 \times S^1$ with Euler number 1. Then $\Sigma^1(X) = S(X) = S^1$, yet $\mathcal{R}^1(X, \mathbb{Q}) = H^1(X, \mathbb{Q}) = \mathbb{Q}^2$.

5.4. **Discussion and examples.** As we shall see in the last few sections, there are several interesting classes of spaces for which the implication from Theorem 5.4 holds as an equivalence. Nevertheless, as the next two examples show, neither the implication from Theorem 5.4, nor the one from Corollary 5.7 can be reversed, in general. We will come back to this point in Example 6.5.

Example 5.10. Consider the 1-relator group $G = \langle x_1, x_2 \mid x_1 x_2 x_1^{-1} = x_2^2 \rangle$. Clearly, $G_{\mathrm{ab}} = \mathbb{Z}$, and so G is 1-formal and $\mathcal{R}^1(G) = \{0\} \subset \mathbb{C}$. A Fox calculus computation shows that $\mathcal{W}^1(G) = \{1, 2\} \subset \mathbb{C}^\times$; thus, $\Omega_1^1(G) = \{\mathrm{pt}\}$, and so $\Omega_1^1(G) = \sigma_1(\mathcal{R}^1(G, \mathbb{Q}))^\complement$. On the other hand, algorithms from [9, 8] show that $\Sigma^1(G) = \{-1\}$, whereas $S(\mathcal{R}^1(G, \mathbb{R}))^\complement = \{\pm 1\}$.

To see why this is the case, consider the abelianization map, $\mathrm{ab}\colon G \twoheadrightarrow \mathbb{Z}$. Then $G' = \ker(\mathrm{ab})$ is isomorphic to $\mathbb{Z}[1/2]$. Hence, $H_1(G', \mathbb{Q}) = \mathbb{Z}[1/2] \otimes \mathbb{Q} = \mathbb{Q}$, which explains why the character ab belongs to $\Omega_1^1(G)$. On the other hand, the group $\mathbb{Z}[1/2]$ is not finitely generated, which explains why $\{\pm\, \mathrm{ab}\} \not\subseteq \Sigma^1(G)$, although $-\,\mathrm{ab} \in \Sigma^1(G)$.

Example 5.11. Consider the space $X = S^1 \vee \mathbb{RP}^2$, with fundamental group $G = \mathbb{Z} * \mathbb{Z}_2$. As before, X is 1-formal and $\mathcal{R}^1(X) = \{0\} \subset \mathbb{C}$. The maximal free abelian cover X^α, corresponding to the projection $\alpha\colon G \twoheadrightarrow \mathbb{Z}$, is homotopy equivalent to a countably infinite wedge of projective planes. Thus, $b_1(X^\alpha) = 0$, and so $\Omega_1^1(X) = \{\mathrm{pt}\}$, which equals $\sigma_1(\mathcal{R}^1(X, \mathbb{Q}))^{\complement}$. On the other hand, since G splits as a non-trivial free product, $\Sigma^1(X, \mathbb{Z}) = \emptyset$, which does not equal $S(\mathcal{R}^1(X, \mathbb{R}))^{\complement} = S^0$.

Finally, here is an example showing how Corollary 5.5 can be used to prove that the inclusions from (38) and (33) are proper, in general. The construction is based on an example of Dwyer and Fried [17], as revisited in more detail in [36].

Example 5.12. Let $Y = T^3 \vee S^2$. Then $\pi_1(Y) = \mathbb{Z}^3$, a free abelian group on generators x_1, x_2, x_3, and $\pi_2(Y) = \mathbb{Z}\mathbb{Z}^3$, a free module generated by the inclusion $S^2 \hookrightarrow Y$. Attaching a 3-cell to Y along a map $S^2 \to Y$ representing the element $x_1 - x_2 + 1$ in $\pi_2(Y)$, we obtain a CW-complex X, with $\pi_1(X) = \mathbb{Z}^3$ and $\pi_2(X) = \mathbb{Z}\mathbb{Z}^3/(x_1 - x_2 + 1)$. Identifying $\widehat{\mathbb{Z}^3} = (\mathbb{C}^\times)^3$, we have that $\mathcal{W}^2(X) = \{t \in (\mathbb{C}^\times)^3 \mid t_1 - t_2 + 1 = 0\}$, and thus $\tau_1(\mathcal{W}^2(X)) = \{0\}$.

Making use of Theorem 3.2, we see that $\Omega_2^2(X)$ consists of precisely two points in $\mathrm{Gr}_2(\mathbb{Q}^3) = \mathbb{QP}^2$; in particular, $\Omega_2^2(X)$ is not a Zariski open subset. Corollary 5.5 now shows that $\Sigma^2(X, \mathbb{Z}) \subsetneq S(\tau_1(\mathcal{W}^2(X)))^{\complement} = S^2$. On the other hand, $\Sigma^2(\mathbb{Z}^3, \mathbb{Z}) = S^2$; thus, $\Sigma^2(X, \mathbb{Z}) \subsetneq \Sigma^2(\pi_1(X), \mathbb{Z})$.

6. Toric complexes

In this section, we illustrate our techniques on a class of spaces that arise in toric topology, as a basic example of polyhedral products. These "toric complexes" are both straight and formal, so it comes as no surprise that both their Ω-invariants and their Σ-invariants are closely related to the resonance varieties.

6.1. **Toric complexes and right-angled Artin groups.** Let L be a simplicial complex with n vertices, and let T^n be the n-torus, with the standard cell decomposition, and with basepoint $*$ at the unique 0-cell.

The *toric complex* associated to L, denoted T_L, is the union of all subcomplexes of the form $T^\sigma = \{x \in T^n \mid x_i = * \text{ if } i \notin \sigma\}$, where σ runs through the simplices of L. Clearly, T_L is a connected CW-complex; its k-cells are in one-to-one correspondence with the $(k-1)$-simplices of L.

Denote by V the set of 0-cells of L, and by E the set of 1-cells of L. The fundamental group, $G_L = \pi_1(T_L)$, is the right-angled Artin group associated to the graph $\Gamma = (\mathsf{V}, \mathsf{E})$, with a generator v for each vertex $v \in \mathsf{V}$, and a commutation relation $vw = wv$ for each edge $\{v, w\} \in \mathsf{E}$.

Much is known about toric complexes and their fundamental groups. For instance, the group G_L has as classifying space the toric complex T_Δ, where $\Delta = \Delta_L$ is the flag complex of L, i.e., the maximal simplicial complex with 1-skeleton equal to that of L. Moreover, the homology groups of T_L are torsion-free, while the cohomology ring

$H^*(T_L, \mathbb{Z})$ is isomorphic to the exterior Stanley-Reisner ring of L, with generators the dual classes $v^* \in H^1(T_L, \mathbb{Z})$, and relations the monomials corresponding to the missing faces of L. Finally, all toric complexes are formal spaces.

For more details and references on all this, we refer to [30, 31, 33, 37].

6.2. **Jump loci and Ω-invariants.** The resonance and characteristic varieties of right-angled Artin groups and toric complexes were studied in [30] and [16], with the complete computation achieved in [31]. We recall here those results, in a form suited for our purposes.

Let $\mathbb{k}$ be a coefficient field. Fixing an ordering on the vertex set V allows us to identify $H^1(T_L, \mathbb{k}^\times)$ with the algebraic torus $(\mathbb{k}^\times)^{\mathsf{V}} = (\mathbb{k}^\times)^n$ and $H^1(T_L, \mathbb{k})$ with the vector space $\mathbb{k}^{\mathsf{V}} = \mathbb{k}^n$. Each subset $\mathsf{W} \subseteq \mathsf{V}$ gives rise to an algebraic subtorus $(\mathbb{k}^\times)^{\mathsf{W}} \subset (\mathbb{k}^\times)^{\mathsf{V}}$ and a coordinate subspace $\mathbb{k}^{\mathsf{W}} \subset \mathbb{k}^{\mathsf{V}}$.

In what follows, we denote by L_{W} the subcomplex induced by L on W, and by $\mathrm{lk}_K(\sigma)$ the link of a simplex $\sigma \in L$ in a subcomplex $K \subseteq L$.

Theorem 6.1 ([31]). *Let L be a simplicial complex on vertex set V. Then, for all $i \geq 1$,*

(44) $$\mathcal{V}^i(T_L, \mathbb{k}) = \bigcup_{\mathsf{W}} (\mathbb{k}^\times)^{\mathsf{W}} \quad and \quad \mathcal{R}^i(T_L, \mathbb{k}) = \bigcup_{\mathsf{W}} \mathbb{k}^{\mathsf{W}},$$

where, in both cases, the union is taken over all subsets $\mathsf{W} \subseteq \mathsf{V}$ for which there is a simplex $\sigma \in L_{\mathsf{V} \setminus \mathsf{W}}$ and an index $j \leq i$ such that $\widetilde{H}_{j-1-|\sigma|}(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma), \mathbb{k}) \neq 0$.

In degree 1, the resonance formula takes a simpler form, already noted in [30]. Clearly, $\mathcal{R}^1(T_L, \mathbb{k})$ depends only on the 1-skeleton $\Gamma = L^{(1)}$; moreover, $\mathcal{R}^1(T_L, \mathbb{k}) = \bigcup_{\mathsf{W}} \mathbb{k}^{\mathsf{W}}$, where the union is taken over all maximal subsets $\mathsf{W} \subseteq \mathsf{V}$ for which the induced graph Γ_{W} is disconnected.

As a consequence of Theorem 6.1, we see that every toric complex is a straight space. Theorem 3.6(2), then, allows us to determine the Dwyer–Fried invariants of such spaces.

Corollary 6.2 ([33], [37]). *Let L be a simplicial complex on vertex set V. Then, for all $i, r \geq 1$,*

(45) $$\Omega_r^i(T_L) = \mathrm{Gr}_r(\mathbb{Q}^{\mathsf{V}}) \setminus \sigma_r(\mathcal{R}^i(T_L, \mathbb{Q})).$$

6.3. **Σ-invariants.** In [3], Bestvina and Brady considered the "diagonal" homomorphism $\nu \colon G_L \to \mathbb{Z}$, $v \mapsto 1$, and the finiteness properties of the corresponding subgroup, $N_L = \ker(\nu)$. One of the main results of [3] determines the maximal integer k for which ν belongs to $\Sigma^k(G_L, \mathbb{Z})$.

The picture was completed by Meier, Meinert, and VanWyk [29] and by Bux and Gonzalez [10], who computed explicitly the Bieri–Neumann–Strebel–Renz invariants of right-angled Artin groups.

Theorem 6.3 ([29, 10]). *Let L be a simplicial complex, and let Δ be the associated flag complex. Let $\chi \in S(G_L)$ be a non-zero homomorphism, with support $\mathsf{W} = \{v \in \mathsf{V} \mid \chi(v) \neq 0\}$, and let $\mathbb{k} = \mathbb{Z}$ or a field. Then, $\chi \in \Sigma^i(G_L, \mathbb{k})$ if and only if*

(46) $$\widetilde{H}_j(\mathrm{lk}_{\Delta_{\mathsf{W}}}(\sigma), \mathbb{k}) = 0,$$

for all $\sigma \in \Delta_{\mathsf{V} \setminus \mathsf{W}}$ and $-1 \leq j \leq i - \dim(\sigma) - 2$.

We would like now to compare the BNSR invariants of a toric complex T_L to the resonance varieties of T_L. Using the fact that toric complexes are straight spaces, Corollary 4.11 gives

(47) $$\Sigma^i(T_L, \mathbb{Z}) \subseteq S(T_L) \setminus S(\mathcal{R}^i(T_L, \mathbb{R})).$$

For right-angled Artin groups, we can say more. Comparing the description of the Σ-invariants of the group G_L given in Theorem 6.3 to that of the resonance varieties of the space $T_\Delta = K(G_L, 1)$ given in Theorem 6.1, yields the following result.

Corollary 6.4 ([33]). *Let G_L be a right-angled Artin group. For each $i \geq 0$, the following hold.*

(1) $\Sigma^i(G_L, \mathbb{R}) = S(\mathcal{R}^i(G_L, \mathbb{R}))^\complement$.
(2) $\Sigma^i(G_L, \mathbb{Z}) = S(\mathcal{R}^i(G_L, \mathbb{R}))^\complement$, *provided that, for every $\sigma \in \Delta$, and every $\mathsf{W} \subseteq \mathsf{V}$ with $\sigma \cap W = \emptyset$, the groups $\widetilde{H}_j(\mathrm{lk}_{\Delta_\mathsf{W}}(\sigma), \mathbb{Z})$ are torsion-free, for all $j \leq i - \dim(\sigma) - 2$.*

The torsion-freeness condition from Corollary 6.4(2) is always satisfied in degree $i = 1$. Thus,

(48) $$\Sigma^1(G_L, \mathbb{Z}) = S(\mathcal{R}^1(G_L, \mathbb{R}))^\complement,$$

an equality already proved (by different methods) in [30]. Nevertheless, the condition is not always satisfied in higher degrees, thus leading to situations where the equality from Corollary 6.4(2) fails. The next example (extracted from [33]) illustrates this phenomenon, while also showing that the implication from Theorem 5.4 cannot always be reversed, even for right-angled Artin groups.

Example 6.5. Let Δ be a flag triangulation of the real projective plane, $\mathbb{RP}^2$, and let $\nu\colon G_\Delta \to \mathbb{Z}$ be the diagonal homomorphism. Then $\nu \notin \Sigma^2(G_\Delta, \mathbb{Z})$, even though $\nu \in \Sigma^2(G_\Delta, \mathbb{R})$. Consequently,

(49) $$\Sigma^2(G_\Delta, \mathbb{Z}) \subsetneqq S(\mathcal{R}^2(G_\Delta, \mathbb{R}))^\complement,$$

although $\Omega_r^2(G_\Delta) = \sigma_r(\mathcal{R}^2(T_\Delta, \mathbb{Q}))^\complement$, for all $r \geq 1$.

7. Quasi-projective varieties

We now discuss the cohomology jumping loci, the Dwyer–Fried invariants, and the Bieri–Neumann–Strebel–Renz invariants of smooth, complex projective and quasi-projective varieties.

7.1. **Complex algebraic varieties.** A smooth, connected manifold X is said to be a *(smooth) quasi-projective variety* if there is a smooth, complex projective variety $\overline{X}$ and a normal-crossings divisor D such that $X = \overline{X} \setminus D$. By a well-known result of Deligne, each cohomology group of a quasi-projective variety X admits a mixed Hodge structure. This puts definite constraints on the topology of such varieties.

For instance, if X admits a non-singular compactification $\overline{X}$ with $b_1(\overline{X}) = 0$, the weight 1 filtration on $H^1(X, \mathbb{C})$ vanishes; in turn, by work of Morgan, this implies the 1-formality of X. Thus, as noted by Kohno, if X is the complement of a hypersurface in $\mathbb{CP}^n$, then $\pi_1(X)$ is 1-formal. In general, though, quasi-projective varieties need not be 1-formal.

If X is actually a (compact, smooth) projective variety, then a stronger statement holds: as shown by Deligne, Griffiths, Morgan, and Sullivan, such a manifold (and,

more generally, a compact Kähler manifold) is formal. In general, though, quasi-projective varieties are not formal, even if their fundamental groups are 1-formal.

Example 7.1. Let $T = E^n$ be the n-fold product of an elliptic curve $E = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}$. The closed form $\frac{1}{2}\sqrt{-1} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i = \sum_{i=1}^{n} dx_i \wedge dy_i$ defines a cohomology class ω in $H^{1,1}(T) \cap H^2(T, \mathbb{Z})$. By the Lefschetz theorem on $(1,1)$-classes (see [25, p. 163]), ω can be realized as the first Chern class of an algebraic line bundle $L \to T$.

Let X be the complement of the zero-section of L; then X is a connected, smooth, quasi-projective variety. Moreover, X deform-retracts onto N, the total space of the circle bundle over the torus $T = (S^1)^{2n}$ with Euler class ω. Clearly, the Heisenberg-type nilmanifold N is not a torus, and thus it is not formal. In fact, as shown by Măcinic in [28, Remark 5.4], the manifold N is $(n-1)$-formal, but not n-formal. Thus, if $n = 1$, the variety X is not 1-formal, whereas if $n > 1$, the variety X is 1-formal, but not formal.

7.2. Cohomology jump loci.

The existence of mixed Hodge structures on the cohomology groups of connected, smooth, complex quasi-projective varieties also puts definite constraints on the nature of their cohomology support loci.

The structure of the characteristic varieties of such spaces (and, more generally, Kähler and quasi-Kähler manifolds) was determined through the work of Beauville, Green and Lazarsfeld, Simpson, Campana, and Arapura in the 1990s. Further improvements and refinements have come through the recent work of Budur, Libgober, Dimca, Artal-Bartolo, Cogolludo, and Matei. We summarize these results, essentially in the form proved by Arapura, but in the simplified (and slightly updated) form we need them here.

Theorem 7.2 ([1]). *Let $X = \overline{X} \setminus D$, where $\overline{X}$ is a smooth, projective variety and D is a normal-crossings divisor.*

(1) *If either $D = \emptyset$ or $b_1(\overline{X}) = 0$, then each characteristic variety $\mathcal{V}^i(X)$ is a finite union of unitary translates of algebraic subtori of $H^1(X, \mathbb{C}^\times)$.*

(2) *In degree $i = 1$, the condition that $b_1(\overline{X}) = 0$ if $D \neq \emptyset$ may be lifted. Furthermore, each positive-dimensional component of $\mathcal{V}^1(X)$ is of the form $\rho \cdot T$, with T an algebraic subtorus, and ρ a torsion character.*

For instance, if C is a connected, smooth complex curve with $\chi(C) < 0$, then $\mathcal{V}^1(C) = H^1(C, \mathbb{C}^\times)$. More generally, if X is a smooth, quasi-projective variety, then every positive-dimensional component of $\mathcal{V}^1(X)$ arises by pullback along a suitable pencil. More precisely, if $\rho \cdot T$ is such a component, then $T = f^*(H^1(C, \mathbb{C}^\times))$, for some curve C, and some holomorphic, surjective map $f \colon X \to C$ with connected generic fiber.

In the presence of 1-formality, the quasi-projectivity of X also imposes stringent conditions on the degree 1 resonance varieties. Theorems 7.2 and 2.8 yield the following characterization of these varieties.

Corollary 7.3 ([16]). *Let X be a 1-formal, smooth, quasi-projective variety. Then $\mathcal{R}^1(X)$ is a finite union of rationally defined linear subspaces of $H^1(X, \mathbb{C})$.*

In fact, much more is proved in [16] about those subspaces. For instance, any two of them intersect only at 0, and the restriction of the cup-product map $H^1(X, \mathbb{C}) \wedge H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C})$ to any one of them has rank equal to either 0 or 1.

Corollary 7.4 ([37])**.** *If X is a 1-formal, smooth, quasi-projective variety, then X is locally 1-straight. Moreover, X is 1-straight if and only if $\mathcal{W}^1(X)$ contains no positive-dimensional translated subtori.*

7.3. **Ω-invariants.** The aforementioned structural results regarding the cohomology jump loci of smooth, quasi-projective varieties inform on the Dwyer–Fried sets of such varieties. For instance, Theorem 7.2 together with Proposition 3.4 yield the following corollary.

Corollary 7.5. *Let $X = \overline{X} \setminus D$ be a smooth, quasi-projective variety with $D = \emptyset$ or $b_1(\overline{X}) = 0$. If $\mathcal{W}^i(X)$ contains no positive-dimensional translated subtori, then $\Omega^i_r(X) = \sigma_r(\tau^{\mathbb{Q}}_1(\mathcal{W}^i(X)))^{\complement}$, for all $r \geq 1$.*

Likewise, Corollary 7.4 together with Theorem 3.6 yield the following corollary.

Corollary 7.6 ([37])**.** *Let X be a 1-formal, smooth, quasi-projective variety. Then:*
  (1) *$\Omega^1_1(X) = \mathbb{P}(\mathcal{R}^1(X, \mathbb{Q}))^{\complement}$ and $\Omega^1_r(X) \subseteq \sigma_r(\mathcal{R}^1(X, \mathbb{Q}))^{\complement}$, for $r \geq 2$.*
  (2) *If $\mathcal{W}^1(X)$ contains no positive-dimensional translated subtori, then $\Omega^1_r(X) = \sigma_r(\mathcal{R}^1(X, \mathbb{Q}))^{\complement}$, for all $r \geq 1$.*

If the characteristic variety $\mathcal{W}^1(X)$ contains positive-dimensional translated components, the resonance variety $\mathcal{R}^1(X, \mathbb{Q})$ may fail to determine all the Dwyer–Fried sets $\Omega^1_r(X)$. This phenomenon is made concrete by the following result.

Theorem 7.7 ([37])**.** *Let X be a 1-formal, smooth, quasi-projective variety. Suppose $\mathcal{W}^1(X)$ has a 1-dimensional component not passing through 1, while $\mathcal{R}^1(X)$ has no codimension-1 components. Then $\Omega^1_2(X)$ is strictly contained in $\mathrm{Gr}_2(H^1(X, \mathbb{Q})) \setminus \sigma_2(\mathcal{R}^1(X, \mathbb{Q}))$.*

7.4. **Σ-invariants.** The cohomology jump loci of smooth, quasi-projective varieties also inform on the Bieri–Neumann–Strebel sets of such varieties. For instance, using Corollary 5.7, we may identify a class of 1-formal, quasi-projective varieties for which inclusion (40) is strict.

Corollary 7.8. *Let X be a 1-formal, smooth, quasi-projective variety. Suppose $\mathcal{W}^1(X)$ has a 1-dimensional component not passing through 1, while $\mathcal{R}^1(X)$ has no codimension-1 components. Then $\Sigma^1(X)$ is strictly contained in $S(X) \setminus S(\mathcal{R}^1(X, \mathbb{R}))$.*

We shall see in Example 9.11 a concrete variety to which this corollary applies.

In the case of smooth, complex projective varieties (or, more generally, compact Kähler manifolds), a different approach is needed in order to show that inclusion (40) may be strict. Indeed, by Theorem 7.2, all components of $\mathcal{W}^1(\overline{X})$ are even-dimensional, so Corollary 7.8 does not apply.

On the other hand, as shown by Delzant in [13], the BNS invariant of a compact Kähler manifold M is determined by the pencils supported by M.

Theorem 7.9 ([13])**.** *Let M be a compact Kähler manifold. Then*

$$(50) \qquad \Sigma^1(M) = S(M) \setminus \bigcup_\alpha S(f^*_\alpha(H^1(C_\alpha, \mathbb{R}))),$$

where the union is taken over those pencils $f_\alpha \colon M \to C_\alpha$ with the property that either $\chi(C_\alpha) < 0$, or $\chi(C_\alpha) = 0$ and f_α has some multiple fiber.

This theorem, together with results from [16], yields the following characterization of those compact Kähler manifolds M for which the inclusion from Corollary 4.12 holds as equality.

Theorem 7.10 ([33]). *Let M be a compact Kähler manifold. Then $\Sigma^1(M) = S(\mathcal{R}^1(M, \mathbb{R}))^{\complement}$ if and only if there is no pencil $f \colon M \to E$ onto an elliptic curve E such that f has multiple fibers.*

7.5. **Examples and discussion.** As noted in [33, Remark 16.6], a general construction due to Beauville [2] shows that equality does not always hold in Theorem 7.10. More precisely, if N is a compact Kähler manifold on which a finite group π acts freely, and $p \colon C \to E$ is a ramified, regular π-cover over an elliptic curve, with at least one ramification point, then the quotient $M = (C \times N)/\pi$ is a compact Kähler manifold admitting a pencil $f \colon M \to E$ with multiple fibers.

An example of this construction is given in [15]. Let C be a Fermat quartic in $\mathbb{CP}^2$, viewed as a 2-fold branched cover of E, and let N be a simply-connected compact Kähler manifold admitting a fixed-point free involution, for instance, a Fermat quartic in $\mathbb{CP}^3$, viewed as a 2-fold unramified cover of the Enriques surface. Then, the Kähler manifold $M = (C \times N)/\mathbb{Z}_2$ admits a pencil with base E, having four multiple fibers, each of multiplicity 2; thus, $\Sigma^1(M, \mathbb{Z}) = \emptyset$. Moreover, direct computation shows that $\mathcal{R}^1(M) = \{0\}$, and so $\Sigma^1(M, \mathbb{Z}) \subsetneq S(\mathcal{R}^1(M, \mathbb{R}))^{\complement}$.

We provide here another example, in the lowest possible dimension, using a complex surface studied by Catanese, Ciliberto, and Mendes Lopes in [12]. For this manifold M, the resonance variety does not vanish, yet $\Sigma^1(M, \mathbb{Z})$ is strictly contained in the complement of $S(\mathcal{R}^1(M, \mathbb{R}))$. We give an alternate explanation of this fact which is independent of Theorems 7.9 and 7.10, but relies instead on results from [38] and on Corollary 5.7.

Example 7.11. Let C_1 be a (smooth, complex) curve of genus 2 with an elliptic involution σ_1 and let C_2 be a curve of genus 3 with a free involution σ_2. Then $\Sigma_1 = C_1/\sigma_1$ is a curve of genus 1, and $\Sigma_2 = C_2/\sigma_2$ is a curve of genus 2.

Now let $M = (C_1 \times C_2)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts via the involution $\sigma_1 \times \sigma_2$. Then M is a smooth, complex projective surface. Projection onto the first coordinate yields a pencil $f_1 \colon M \to \Sigma_1$ with two multiple fibers, each of multiplicity 2, while projection onto the second coordinate defines a smooth fibration $f_2 \colon M \to \Sigma_2$. By Theorem 7.10, we have that $\Sigma^1(M, \mathbb{Z}) \subsetneq S(\mathcal{R}^1(M, \mathbb{R}))^{\complement}$.

Here is an alternate explanation. Using the fact that $H_1(M, \mathbb{Z}) = \mathbb{Z}^6$, we may identify $H^1(M, \mathbb{C}^\times) = (\mathbb{C}^\times)^6$. In [38], we showed that

(51) $$\mathcal{V}^1(M) = \{t_4 = t_5 = t_6 = 1, \ t_3 = -1\} \cup \{t_1 = t_2 = 1\},$$

with the two components corresponding to the pencils f_1 and f_2, respectively, from which we inferred that the set $\Omega_2^1(M)$ is not open, not even in the usual topology on $\mathrm{Gr}_2(\mathbb{Q}^6)$. Now, from (51), we also see that $\mathcal{R}^1(M) = \{x_1 = x_2 = 0\}$. Since $\Omega_2^1(M)$ is not open, it must be a proper subset of $\sigma_2(\mathcal{R}^1(M, \mathbb{Q}))^{\complement}$. In view of Corollary 5.7, we conclude once again that $\Sigma^1(M, \mathbb{Z}) \subsetneq S(\mathcal{R}^1(M, \mathbb{R}))^{\complement}$.

8. CONFIGURATION SPACES

We now consider in more detail a particularly interesting class of quasi-projective varieties, obtained by deleting the "fat diagonal" from the n-fold Cartesian product of a smooth, complex algebraic curve.

8.1. **Ordered configurations on algebraic varieties.** A construction due to Fadell and Neuwirth associates to a space X and a positive integer n the space of ordered configurations of n points in X,

(52) $$F(X, n) = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

The most basic example is the configuration space of n ordered points in $\mathbb{C}$, which is a classifying space for P_n, the pure braid group on n strings, whose cohomology ring was computed by Arnol'd in the late 1960s.

The E_2-term of the Leray spectral sequence for the inclusion $F(X, n) \hookrightarrow X^n$ was described concretely by Cohen and Taylor in the late 1970s. If X is a smooth, complex projective variety of dimension m, then, as shown by Totaro in [39], the Cohen–Taylor spectral sequence collapses at the E_{m+1}-term, and $H^*(F(X, n), \mathbb{C}) = E_{m+1}$, as graded algebras.

Particularly interesting is the case of a Riemann surface Σ_g. The ordered configuration space $F(\Sigma_g, n)$ is a classifying space for $P_n(\Sigma_g)$, the pure braid group on n strings of the underlying surface. In [4], Bezrukavnikov gave explicit presentations for the Malcev Lie algebras $\mathfrak{m}_{g,n} = \mathfrak{m}(P_n(\Sigma_g))$, from which he concluded that the pure braid groups on surfaces are 1-formal for $g > 1$ or $g = 1$ and $n \leq 2$, but not 1-formal for $g = 1$ and $n \geq 3$. The non-1-formality of the groups $P_n(\Sigma_1)$, $n \geq 3$, is also established in [16], by showing that the tangent cone formula (18) fails in this situation (see Example 8.2 below for the case $n = 3$).

Remark 8.1. In [11, Proposition 5], Calaque, Enriquez, and Etingof prove that $P_n(\Sigma_1)$ is formal, for all $n \geq 1$. But the notion of formality that these authors use is weaker than the usual notion of 1-formality: their result is that $\mathfrak{m}_{1,n}$ is isomorphic as a filtered Lie algebra with the completion (with respect to the bracket length filtration) of the associated graded Lie algebra, $\mathrm{gr}(\mathfrak{m}_{1,n})$. The failure of 1-formality comes from the fact that $\mathrm{gr}(\mathfrak{m}_{1,n})$ is *not* a quadratic Lie algebra, for $n \geq 3$.

8.2. **Ordered configurations on the torus.** For the reasons outlined above, it makes sense to look more carefully at the configuration spaces of an elliptic curve Σ_1. The resonance varieties $\mathcal{R}^1(F(\Sigma_1, n))$ were computed in [16], while the positive-dimensional components of $\mathcal{V}^1(F(\Sigma_1, n))$ were determined in [14].

Since $\Sigma_1 = S^1 \times S^1$ is a topological group, the space $F(\Sigma_1, n)$ splits up to homeomorphism as a direct product, $F(\Sigma_1', n - 1) \times \Sigma_1$, where Σ_1' denotes Σ_1 with the identity removed. Thus, for all practical purposes, it is enough to consider the space $F(\Sigma_1', n - 1)$. For the sake of concreteness, we will work out in detail the case $n = 3$; the general case may be treated similarly.

Example 8.2. Let $X = F(\Sigma_1', 2)$ be the configuration space of 2 labeled points on a punctured torus. The cohomology ring of X is the exterior algebra on generators a_1, a_2, b_1, b_2 in degree 1, modulo the ideal spanned by the forms $a_1 b_2 + a_2 b_1$, $a_1 b_1$, and $a_2 b_2$. The first resonance variety is an irreducible quadric hypersurface in $\mathbb{C}^4$, given by

$$\mathcal{R}^1(X) = \{x_1 y_2 - x_2 y_1 = 0\}.$$

Corollary 7.3, then, shows that X is not 1-formal.

The first characteristic variety of X consists of three 2-dimensional algebraic subtori of $(\mathbb{C}^\times)^4$:

$$\mathcal{V}^1(X) = \{t_1 = t_2 = 1\} \cup \{s_1 = s_2 = 1\} \cup \{t_1 s_1 = t_2 s_2 = 1\}.$$

These three subtori arise from the fibrations $F(\Sigma_1', 2) \to \Sigma_1'$ obtained by sending a point (z_1, z_2) to z_1, z_2, and $z_1 z_2^{-1}$, respectively. It follows that $\tau_1(\mathcal{V}^1(X)) = \mathrm{TC}_1(\mathcal{V}^1(X))$, but both types of tangent cones are properly contained in the resonance variety $\mathcal{R}^1(X)$. Moreover, the characteristic subspace arrangement $\mathcal{C}_1(X)$ consists of three, pairwise transverse planes in $\mathbb{Q}^4$, namely,

$$L_1 = \{x_1 = x_2 = 0\}, \ L_2 = \{y_1 = y_2 = 0\}, \ L_3 = \{x_1 + y_1 = x_2 + y_2 = 0\}.$$

By Proposition 3.4, the Dwyer–Fried sets $\Omega_r^1(X)$ are obtained by removing from $\mathrm{Gr}_r(\mathbb{Q}^4)$ the Schubert varieties $\sigma_r(L_1)$, $\sigma_r(L_2)$, and $\sigma_r(L_3)$. We treat each rank r separately.

- When $r = 1$, the set $\Omega_1^1(X)$ is the complement in $\mathbb{QP}^3$ of the three projective lines defined by L_1, L_2, and L_3.
- When $r = 2$, the Grassmannian $\mathrm{Gr}_2(\mathbb{Q}^4)$ is the quadric hypersurface in $\mathbb{QP}^5$ given in Plücker coordinates by the equation $p_{12}p_{34} - p_{13}p_{24} + p_{23}p_{14} = 0$. The set $\Omega_2^1(X)$, then, is the complement in $\mathrm{Gr}_2(\mathbb{Q}^4)$ of the variety cut out by the hyperplanes $p_{12} = 0$, $p_{34} = 0$, and $p_{12} - p_{23} + p_{14} + p_{34} = 0$.
- When $r \geq 3$, the set $\Omega_r^1(X)$ is empty.

Finally, by Corollary 4.10, the BNS set $\Sigma^1(X, \mathbb{Z})$ is included in the complement in S^3 of the three great circles cut out by the real planes spanned by L_1, L_2, and L_3, respectively. It would be interesting to know whether this inclusion is actually an equality.

9. Hyperplane arrangements

We conclude with another interesting class of quasi-projective varieties, obtained by deleting finitely many hyperplanes from a complex affine space.

9.1. Complement and intersection lattice.
A (central) hyperplane arrangement $\mathcal{A}$ is a finite collection of codimension 1 linear subspaces in a complex affine space $\mathbb{C}^\ell$. A defining polynomial for $\mathcal{A}$ is the product $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$, where $\alpha_H \colon \mathbb{C}^\ell \to \mathbb{C}$ is a linear form whose kernel is H.

The main topological object associated to an arrangement is its complement, $X(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$. This is a connected, smooth, quasi-projective variety, whose homotopy-type invariants are intimately tied to the combinatorics of the arrangement. The latter is encoded in the intersection lattice, $L(\mathcal{A})$, which is the poset of all (non-empty) intersections of $\mathcal{A}$, ordered by reverse inclusion. The rank of the arrangement, denoted $\mathrm{rk}(\mathcal{A})$, is the codimension of $\bigcap_{H \in \mathcal{A}} H$.

Example 9.1. A familiar example is the rank $\ell - 1$ braid arrangement, consisting of the diagonal hyperplanes $H_{ij} = \{z_i - z_j = 0\}$ in $\mathbb{C}^\ell$. The complement is the configuration space $F(\mathbb{C}, \ell)$, while the intersection lattice is the lattice of partitions of $\{1, \ldots, \ell\}$, ordered by refinement.

For a general arrangement $\mathcal{A}$, the cohomology ring of the complement was computed by Brieskorn in the early 1970s, building on work of Arnol'd on the cohomology ring of the braid arrangement. It follows from Brieskorn's work that the space $X(\mathcal{A})$ is formal. In 1980, Orlik and Solomon gave a simple combinatorial description of the ring $H^*(X(\mathcal{A}), \mathbb{Z})$: it is the quotient of the exterior algebra on degree-one classes

e_H dual to the meridians around the hyperplanes $H \in \mathcal{A}$, modulo a certain ideal (generated in degrees greater than one) determined by the intersection lattice.

Let $\overline{\mathcal{A}} = \{\mathbb{P}(H)\}_{H \in \mathcal{A}}$ be the projectivization of $\mathcal{A}$, and let $X(\overline{\mathcal{A}})$ be its complement in $\mathbb{CP}^{\ell-1}$. The standard $\mathbb{C}^{\times}$-action on $\mathbb{C}^{\ell}$ restricts to a free action on $X(\mathcal{A})$; the resulting fiber bundle, $\mathbb{C}^{\times} \to X(\mathcal{A}) \to X(\overline{\mathcal{A}})$, is readily seen to be trivial. Under the resulting identification, $X(\mathcal{A}) = X(\overline{\mathcal{A}}) \times \mathbb{C}^{\times}$, the group $H^1(\mathbb{C}^{\times}, \mathbb{Z}) = \mathbb{Z}$ is spanned by the vector $\sum_{H \in \mathcal{A}} e_H \in H^1(X(\mathcal{A}), \mathbb{Z})$.

9.2. Cohomology jump loci.

The resonance varieties $\mathcal{R}^i(\mathcal{A}) := \mathcal{R}^i(X(\mathcal{A}), \mathbb{C})$ were first defined and studied by Falk in [19]. Clearly, these varieties depend only on the graded ring $H^*(X(\mathcal{A}), \mathbb{C})$, and thus, only on the intersection lattice $L(\mathcal{A})$.

Now fix a linear ordering on the hyperplanes of $\mathcal{A}$, and identify $H^1(X(\mathcal{A}), \mathbb{C}) = \mathbb{C}^n$, where $n = |\mathcal{A}|$. From the product formula (17) for resonance varieties (or from an old result of Yuzvinsky [40]), we see that $\mathcal{R}^i(\mathcal{A})$ is isomorphic to $\mathcal{R}^i(\overline{\mathcal{A}})$, and lies in the hyperplane $x_1 + \cdots + x_n = 0$ inside $\mathbb{C}^n$. Similarly, the characteristic varieties $\mathcal{V}^i(\mathcal{A}) := \mathcal{V}^i(X(\mathcal{A}), \mathbb{C})$ lie in the subtorus $t_1 \cdots t_n = 1$ inside the complex algebraic torus $H^1(X(\mathcal{A}), \mathbb{C}^{\times}) = (\mathbb{C}^{\times})^n$.

In view of the Lefschetz-type theorem of Hamm and Lê, taking a generic two-dimensional section does not change the fundamental group of the complement. Thus, in order to describe the variety $\mathcal{R}^1(\mathcal{A})$, we may assume $\mathcal{A}$ is a affine arrangement of n lines in $\mathbb{C}^2$, for which no two lines are parallel.

The structure of the first resonance variety of an arrangement was worked out in great detail in work of Cohen, Denham, Falk, Libgober, Pereira, Suciu, Yuzvinsky, and many others. It is known that each component of $\mathcal{R}^1(\mathcal{A})$ is a linear subspace in $\mathbb{C}^n$, while any two distinct components meet only at 0. The simplest components of the resonance variety are those corresponding to multiple points of $\mathcal{A}$: if m lines meet at a point, then $\mathcal{R}^1(\mathcal{A})$ acquires an $(m-1)$-dimensional linear subspace. The remaining components (of dimension either 2 or 3), correspond to certain "neighborly partitions" of sub-arrangements of $\mathcal{A}$.

Example 9.2. Let $\mathcal{A}$ be a generic 3-slice of the braid arrangement of rank 3, with defining polynomial $Q(\mathcal{A}) = z_0 z_1 z_2 (z_0 - z_1)(z_0 - z_2)(z_1 - z_2)$. Take a generic plane section, and label the corresponding lines as $1, \ldots, 6$. Then, the variety $\mathcal{R}^1(\mathcal{A}) \subset \mathbb{C}^6$ has 4 local components, corresponding to the triple points $124, 135, 236, 456$, and one non-local component, corresponding to the neighborly partition $(16|25|34)$.

From Theorem 7.2, we know that $\mathcal{V}^1(\mathcal{A})$ consists of subtori in $(\mathbb{C}^{\times})^n$, possibly translated by roots of unity, together with a finite number of torsion points. By Theorem 2.8—first proved in the context of hyperplane arrangements by Cohen–Suciu and Libgober—we have that $\text{TC}_1(\mathcal{V}^1(\mathcal{A})) = \mathcal{R}^1(\mathcal{A})$. Thus, the components of $\mathcal{V}^1(\mathcal{A})$ passing through the origin are completely determined by $\mathcal{R}^1(\mathcal{A})$, and hence, by $L(\mathcal{A})$: to each linear subspace L in $\mathcal{R}^1(\mathcal{A})$ there corresponds an algebraic subtorus, $T = \exp(L)$, in $\mathcal{V}^1(\mathcal{A})$.

As pointed out in [35], though, the characteristic variety $\mathcal{V}^1(\mathcal{A})$ may contain translated subtori—that is, components not passing through 1. Despite much work since then, it is still not known whether such components are combinatorially determined.

9.3. **Upper bounds for the Ω- and Σ-sets.** We are now ready to consider the Dwyer–Fried and the Bieri–Neumann–Strebel–Renz invariants associated to a hyperplane arrangement $\mathcal{A}$. For simplicity of notation, we will write

$$(53) \qquad \Omega_r^i(\mathcal{A}) := \Omega_r^i(X(\mathcal{A})),$$

and view this set as lying in the Grassmannian $\mathrm{Gr}_r(\mathcal{A}) := \mathrm{Gr}_r(H^1(X(\mathcal{A}), \mathbb{Q}))$ of r-planes in a rational vector space of dimension $n = |\mathcal{A}|$, with a fixed basis given by the meridians around the hyperplanes. Similarly, we will write

$$(54) \qquad \Sigma^i(\mathcal{A}) := \Sigma^i(X(\mathcal{A}), \mathbb{Z}),$$

and view this set as an open subset inside the $(n-1)$-dimensional sphere $S(\mathcal{A}) = S(H^1(X(\mathcal{A}), \mathbb{R}))$.

As noted in [33, 37], it follows from work of Arapura [1] and Esnault, Schechtman and Viehweg [18] that every arrangement complement is locally straight. In view of Theorem 3.6(1) and Corollary 4.11, then, we have the following corollaries.

Corollary 9.3 ([37]). *For all $i \geq 1$ and $r \geq 1$,*

$$(55) \qquad \Omega_r^i(\mathcal{A}) \subseteq \mathrm{Gr}_r(\mathcal{A}) \setminus \sigma_r(\mathcal{R}^i(X(\mathcal{A}), \mathbb{Q})).$$

Corollary 9.4 ([33]). *For all $i \geq 1$,*

$$(56) \qquad \Sigma^i(\mathcal{A}) \subseteq S(\mathcal{A}) \setminus S(\mathcal{R}^i(X(\mathcal{A}), \mathbb{R})).$$

Since the resonance varieties of $\mathcal{A}$ depend only on its intersection lattice, these upper bounds for the Ω- and Σ-invariants are combinatorially determined. Furthermore, Corollary 5.6 yields the following.

Corollary 9.5. *Suppose $\Sigma^i(\mathcal{A}) = S(\mathcal{A}) \setminus S(\mathcal{R}^i(X(\mathcal{A}), \mathbb{R}))$. Then $\Omega_r^i(\mathcal{A}) = \mathrm{Gr}_r(\mathcal{A}) \setminus \sigma_r(\mathcal{R}^i(X(\mathcal{A}), \mathbb{Q}))$, for all $r \geq 1$.*

9.4. **Lower bounds for the Σ-sets.** In this context, the following recent result of Kohno and Pajitnov [26] is relevant.

Theorem 9.6 ([26]). *Let $\mathcal{A}$ be an arrangement of n hyperplanes, and let $\chi = (\chi_1, \ldots, \chi_n)$ be a vector in $S^{n-1} = S(\mathcal{A})$, with all components χ_j strictly positive. Then $H_i(X, \widehat{\mathbb{Z}G_\chi}) = 0$, for all $i < \mathrm{rk}(\mathcal{A})$.*

Let χ be a positive vector as above. Making use of Theorem 4.8, we infer that $-\chi \in \Sigma^i(\mathcal{A})$, for all $i < \mathrm{rk}(\mathcal{A})$. On the other hand, since $\sum_{j=1}^n \chi_j \neq 0$, we also have that $-\chi \notin S(\mathcal{R}^i(X(\mathcal{A}), \mathbb{R}))$, a fact predicted by Corollary 9.4.

Denote by $S^-(\mathcal{A}) = S^{n-1} \cap (\mathbb{R}_{<0})^n$ the negative octant in the unit sphere $S(\mathcal{A})$. In view of the above discussion, Theorem 4.8 provides the following lower bound for the BNSR invariants of arrangements.

Corollary 9.7. *Let $\mathcal{A}$ be a (central) hyperplane arrangement. Then $S^-(\mathcal{A}) \subset \Sigma^i(\mathcal{A})$, for all $i < \mathrm{rk}(\mathcal{A})$. In particular, $S^-(\mathcal{A}) \subset \Sigma^1(\mathcal{A})$.*

The above lower bound for the BNS invariant of an arrangement $\mathcal{A}$ can be improved quite a bit, by considering the projectivized arrangement $\overline{\mathcal{A}}$. Set $n = |\mathcal{A}|$, and identify the unit sphere $S(\overline{\mathcal{A}}) = S(H^1(X(\overline{\mathcal{A}}), \mathbb{R}))$ with the great sphere $S^{n-2} = \{\chi \in S^{n-1} \mid \sum_{i=1}^n \chi_j = 0\}$ inside $S^{n-1} = S(\mathcal{A})$. Clearly, $S^-(\mathcal{A}) \subset S(\mathcal{A}) \setminus S(\overline{\mathcal{A}})$.

Proposition 9.8. *For any central arrangement $\mathcal{A}$,*

$$(57) \qquad\qquad \Sigma^1(\mathcal{A}) = S(\mathcal{A}) \setminus (S(\overline{\mathcal{A}}) \setminus \Sigma^1(\overline{\mathcal{A}})).$$

In particular, $S(\mathcal{A}) \setminus S(\overline{\mathcal{A}}) \subseteq \Sigma^1(\mathcal{A})$.

Proof. By the product formula (36) for the BNS invariants, we have that $\Sigma^1(\mathcal{A})^{\complement} = \Sigma^1(\overline{\mathcal{A}})^{\complement}$. The desired conclusion follows. $\qquad\square$

9.5. **Discussion and examples.** In simple situations, the Dwyer–Fried and Bieri–Neumann–Strebel invariants of an arrangement can be computed explicitly, and the answers agree with those predicted by the upper bounds from §9.3.

Example 9.9. Let $\mathcal{A}$ be a pencil of $n \geq 3$ lines through the origin of $\mathbb{C}^2$. Then $X(\mathcal{A})$ is diffeomorphic to $\mathbb{C}^{\times} \times (\mathbb{C} \setminus \{n-1 \text{ points}\})$, which in turn is homotopy equivalent to the toric complex T_L, where $L = K_{1,n-1}$ is the bipartite graph obtained by coning a discrete set of $n-1$ points. Thus, the Ω- and Σ-invariants of $\mathcal{A}$ can be computed using the formulas from §6. For instance, $\mathcal{R}^1(\mathcal{A}) = \mathbb{C}^{n-1} \subset \mathbb{C}^n$. Therefore, $\Omega_1^1(\mathcal{A}) = \mathbb{QP}^{n-1} \setminus \mathbb{QP}^{n-2}$ and $\Omega_2^1(\mathcal{A}) = \emptyset$. Moreover, $\Sigma^1(\mathcal{A}) = S^{n-1} \setminus S^{n-2}$, which is the same as the lower bound from Proposition 9.8.

For arbitrary arrangements, the computation of the Ω- and Σ-invariants is far from being done, even in degree $i = 1$. A more detailed analysis of the Ω-invariants of arrangements is given in [37, 38]. Here is a sample result.

Proposition 9.10 ([37]). *Let $\mathcal{A}$ be an arrangement of n lines in $\mathbb{C}^2$. Suppose $\mathcal{A}$ has 1 or 2 lines which contain all the intersection points of multiplicity 3 and higher. Then $\Omega_r^1(\mathcal{A}) = \mathrm{Gr}_r(\mathbb{Q}^n) \setminus \sigma_r(\mathcal{R}^1(X(\mathcal{A}), \mathbb{Q}))$, for all $r \geq 1$.*

The reason is that, by a result of Nazir and Raza, the first characteristic variety of such an arrangement has no translated components, and so $X(\mathcal{A})$ is 1-straight. In general, though, translated tori in the characteristic variety may affect both the Ω-sets and the Σ-sets of the arrangement.

Example 9.11. Let $\mathcal{A}$ be the deleted B$_3$ arrangement, with defining polynomial $Q(\mathcal{A}) = z_0 z_1 (z_0^2 - z_1^2)(z_0^2 - z_2^2)(z_1^2 - z_2^2)$. The jump loci of this arrangement were computed in [35]. The resonance variety $\mathcal{R}^1(\mathcal{A}) \subset \mathbb{C}^8$ contains 7 local components, corresponding to 6 triple points and one quadruple point, and 5 other components, corresponding to braid sub-arrangements. In particular, $\mathrm{codim}\,\mathcal{R}^1(\mathcal{A}) = 5$. In addition to the 12 subtori arising from the subspaces in $\mathcal{R}^1(\mathcal{A})$, the characteristic variety $\mathcal{V}^1(\mathcal{A}) \subset (\mathbb{C}^{\times})^8$ also contains a component of the form $\rho \cdot T$, where T is a 1-dimensional algebraic subtorus, and ρ is a root of unity of order 2.

Of course, the complement of $\mathcal{A}$ is a formal, smooth, quasi-projective variety. From Theorem 7.7, we deduce that the Dwyer–Fried set $\Omega_2^1(\mathcal{A})$ is strictly contained in $\sigma_2(\mathcal{R}^1(X(\mathcal{A}), \mathbb{Q}))^{\complement}$. Using Corollary 9.5, we conclude that the BNS set $\Sigma^1(\mathcal{A})$ is strictly contained in $S(\mathcal{R}^1(X(\mathcal{A}), \mathbb{R}))^{\complement}$.

This example answers in the negative Question 9.18(ii) from [36]. It would be interesting to compute explicitly the Ω-invariants and Σ-invariants of wider classes of arrangements, and see whether these invariants depend only on the intersection lattice, or also on other, more subtle data.

10. Acknowledgements

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