

# QUASI-KÄHLER BESTVINA-BRADY GROUPS

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## Abstract

A finite simple graph  $\Gamma$  determines a right-angled Artin group  $G_\Gamma$ , with one generator for each vertex  $v$ , and with one commutator relation  $vw = wv$  for each pair of vertices joined by an edge. The Bestvina-Brady group  $N_\Gamma$  is the kernel of the projection  $G_\Gamma \rightarrow \mathbb{Z}$ , which sends each generator  $v$  to 1. We establish precisely which graphs  $\Gamma$  give rise to quasi-Kähler (respectively, Kähler) groups  $N_\Gamma$ . This yields examples of quasi-projective groups which are not commensurable (up to finite kernels) to the fundamental group of any aspherical, quasi-projective variety.

## 1. Introduction

**1.1.** Every finitely presented group  $G$  is the fundamental group of a smooth, compact, connected manifold  $M$  of dimension 4 or higher. Requiring that  $M$  be a complex projective manifold (or, more generally, a compact Kähler manifold), puts extremely strong restrictions on what  $G = \pi_1(M)$  can be; see [1]. Groups arising in this fashion are called projective (respectively, Kähler) groups. It is an open question whether Kähler groups need be projective.

A related question, due to J.-P. Serre, asks: which finitely presented groups can be realized as fundamental groups of complements of normal crossing divisors in smooth, connected, complex projective varieties? Groups arising in this fashion are called quasi-projective groups, while groups that can be realized as fundamental groups of complements of normal crossing divisors in compact Kähler manifolds are called quasi-Kähler groups. Again, it is an open question whether quasi-Kähler groups need be quasi-projective.

Finally, we have the following question raised by J. Kollár in [17], section 0.3.1: given a (quasi-) projective group  $G$ , is there a group  $\pi$ , commensurable to  $G$  up to finite kernels, and admitting a  $K(\pi, 1)$  which is a quasi-projective variety? (Two groups are commensurable up to finite kernels if there exists a zig-zag of homomorphisms, each with finite kernel and cofinite image, connecting one group to the other. See Remark 4.4 for more details.)

**1.2.** To a finite simple graph  $\Gamma = (\mathbf{V}, \mathbf{E})$ , there is associated a right-angled Artin group,  $G_\Gamma$ , with a generator  $v$  for each vertex  $v \in \mathbf{V}$ , and with a commutator relation  $vw = wv$  for each edge  $\{v, w\} \in \mathbf{E}$ . The groups  $G_\Gamma$  interpolate between the free groups  $F_n$  (corresponding to the discrete graphs on  $n$  vertices,  $\overline{K}_n$ ), and the free abelian groups  $\mathbb{Z}^n$  (corresponding to the complete

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graphs on  $n$  vertices,  $K_n$ ). These groups are completely determined by their underlying graphs:  $G_\Gamma \cong G_{\Gamma'}$  if and only if  $\Gamma \cong \Gamma'$ ; see [16, 12].

In a previous paper [11], we settled Serre's question for the class of right-angled Artin groups:  $G_\Gamma$  is quasi-projective iff  $G_\Gamma$  is quasi-Kähler iff  $\Gamma$  is a complete multipartite graph (i.e., a join  $K_{n_1, \dots, n_r} = \overline{K}_{n_1} * \dots * \overline{K}_{n_r}$  of discrete graphs). The object of this note is to answer Serre's question for the closely related class of Bestvina-Brady groups. We also answer Kollár's question, in the context of quasi-projective groups.

**1.3.** Let  $\nu: G_\Gamma \rightarrow \mathbb{Z}$  be the homomorphism which sends each generator  $v \in V$  to 1. The Bestvina-Brady group (or, Artin kernel) associated to  $\Gamma$  is defined as  $N_\Gamma = \ker(\nu)$ . Unlike right-angled Artin groups, the groups  $N_\Gamma$  are *not* classified by the underlying graphs  $\Gamma$ . Moreover, these groups are complicated enough that a counterexample to either the Eilenberg-Ganea conjecture or the Whitehead conjecture can be constructed from them, according to the fundamental paper of Bestvina and Brady [3].

Our answer to Serre's problem for this class of groups is as follows.

**Theorem 1.1.** *Let  $\Gamma$  be a finite simplicial graph, with associated Bestvina-Brady group  $N_\Gamma$ . The following are equivalent:*

- (Q1) *The group  $N_\Gamma$  is quasi-Kähler.*
- (Q2) *The group  $N_\Gamma$  is quasi-projective.*
- (Q3) *The graph  $\Gamma$  is either a tree, or a complete multipartite graph  $K_{n_1, \dots, n_r}$ , with either some  $n_i = 1$ , or all  $n_i \geq 2$  and  $r \geq 3$ .*

Implication (Q2)  $\Rightarrow$  (Q1) is clear. Implication (Q1)  $\Rightarrow$  (Q3) is established in Section 4; the proof is based on certain cohomological obstructions to realizability by quasi-Kähler manifolds, developed in [11], and on computations of algebraic invariants for Bestvina-Brady groups, done in [21]. Implication (Q3)  $\Rightarrow$  (Q2) is established in Section 5; the proof is based on a result from [10] on the topology of the mapping  $f: \mathbb{C}^n \rightarrow \mathbb{C}$  induced by the defining equation of an affine hyperplane arrangement in  $\mathbb{C}^n$ .

**1.4.** As a consequence, we can classify the Bestvina-Brady groups arising as fundamental groups of quasi-Kähler manifolds.

**Corollary 1.2.** *The class of quasi-Kähler Bestvina-Brady groups equals the union of the following disjoint classes (which contain no repetitions):*

- (1)  $\mathbb{Z}^r$ , with  $r \geq 0$ ;
- (2)  $\prod_{i=1}^s F_{n_i}$ , with all  $n_i > 1$ ;
- (3)  $\mathbb{Z}^r \times \prod_{i=1}^s F_{n_i}$ , with  $r > 0$  and all  $n_i > 1$ ;
- (4)  $N_{K_{n_1, \dots, n_r}}$ , with all  $n_i \geq 2$  and  $r \geq 3$ .

In the Kähler case, the above theorem takes the following, simpler form.

**Corollary 1.3.** *Let  $\Gamma$  be a finite simplicial graph, with associated Bestvina-Brady group  $N_\Gamma$ . The following are equivalent:*

- (K1) *The group  $N_\Gamma$  is Kähler.*
- (K2) *The group  $N_\Gamma$  is projective.*
- (K3) *The graph  $\Gamma$  is a complete graph  $K_n$  with  $n$  odd.*
- (K4) *The group  $N_\Gamma$  is free abelian of even rank.*

Our answer to Kollár's question, in the context of quasi-projective Bestvina-Brady groups, is as follows.

**Corollary 1.4.** *Let  $N_\Gamma$  be a quasi-projective (equivalently, quasi-Kähler) Bestvina-Brady group. Then, in terms of the classification from Corollary 1.2:*

- (i)  $N_\Gamma$  is of type (1), (2), or (3) if and only if it is the fundamental group of an aspherical, smooth quasi-projective variety.
- (ii)  $N_\Gamma$  is of type (4) if and only if it is not commensurable up to finite kernels to the fundamental group of any aspherical, quasi-projective variety.

The simplest example of a group of type (ii) is the group  $G = N_{K_{2,2,2}}$ , already studied by Stallings in [23]. As noted in [20], this group is the fundamental group of the complement of an arrangement of lines in  $\mathbb{C}\mathbb{P}^2$ ; thus,  $G$  is quasi-projective. On the other hand, as shown by Stallings,  $H_3(G, \mathbb{Z})$  is not finitely generated; thus, there is no  $K(G, 1)$  space with finite 3-skeleton.

The three corollaries above are proved at the end of Section 4.

## 2. Resonance obstructions

We start by reviewing the cohomological obstructions to realizability by quasi-Kähler manifolds that we will need in the sequel.

Let  $G$  be a finitely presented group. Denote by  $A = H^*(G, \mathbb{C})$  the cohomology algebra of  $G$ . For each  $a \in A^1$ , we have  $a^2 = 0$ , and so right-multiplication by  $a$  defines a cochain complex  $(A, a): A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2$ . Let  $\mathcal{R}_1(G)$  be the set of points  $a \in A^1$  where this complex fails to be exact,

$$\mathcal{R}_1(G) = \{a \in A^1 \mid H^1(A, a) \neq 0\}.$$

The set  $\mathcal{R}_1(G)$  is a homogeneous algebraic variety in the affine space  $A^1 = H^1(G, \mathbb{C})$ , called the *(first) resonance variety* of  $G$ .

The group  $G$  is said to be *1-formal* if its Malcev Lie algebra is quadratically presented. This important notion goes back to the work of Quillen [22] and Sullivan [24] on rational homotopy theory; for a comprehensive account and recent results on 1-formal groups, see [18].

The best-known examples of 1-formal groups are the Kähler groups, see [8]. Other classes of examples include Artin groups [15] and finitely-presented Bestvina-Brady groups [21]. The 1-formality property is preserved under free products and direct products, see [11].

**Theorem 2.1** ([11]). *Let  $M$  be a quasi-Kähler manifold. Set  $G = \pi_1(M)$  and let  $\mathcal{R}_1(G) = \bigcup_\alpha \mathcal{R}^\alpha$  be the decomposition of the first resonance variety of  $G$  into irreducible components. If  $G$  is 1-formal, then the following hold.*

- (1) *Every positive-dimensional component  $\mathcal{R}^\alpha$  is a  $p$ -isotropic linear subspace of  $H^1(G, \mathbb{C})$ , of dimension at least  $2p + 2$ , for some  $p = p(\alpha) \in \{0, 1\}$ .*
- (2) *If  $M$  is Kähler, then only 1-isotropic components can occur.*

Here, we say that a nonzero subspace  $V \subseteq H^1(G, \mathbb{C})$  is 0- (respectively, 1-) *isotropic* if the restriction of the cup-product map,  $\cup_G: \bigwedge^2 V \rightarrow \cup_G(\bigwedge^2 V)$ , is equivalent to  $\cup_C: \bigwedge^2 H^1(C, \mathbb{C}) \rightarrow H^2(C, \mathbb{C})$ , where  $C$  is a non-compact (respectively, compact) smooth, connected complex curve.

### 3. Cohomology ring and resonance varieties for $G_\Gamma$ and $N_\Gamma$

Let  $\Gamma = (\mathbf{V}, \mathbf{E})$  be a finite simple graph, with  $G_\Gamma = \langle v \in \mathbf{V} \mid vw = wv \text{ if } \{v, w\} \in \mathbf{E} \rangle$  the corresponding right-angled Artin group, and  $N_\Gamma = \ker(\nu: G_\Gamma \rightarrow \mathbb{Z})$  the corresponding Bestvina-Brady group. In this section, we review some facts about the cohomology in low degrees and the resonance varieties for the groups  $G_\Gamma$  and  $N_\Gamma$ .

**3.1.** Write  $H_\mathbf{V} = H^1(G_\Gamma, \mathbb{C})$ . If  $\mathbf{W}$  is a subset of  $\mathbf{V}$ , write  $\Gamma_\mathbf{W}$  for the induced subgraph on vertex set  $\mathbf{W}$ , and  $H_\mathbf{W}$  for the corresponding coordinate subspace of  $H_\mathbf{V}$ .

**Theorem 3.1** ([19]). *Let  $\Gamma$  be a finite graph. Then:*

$$\mathcal{R}_1(G_\Gamma) = \bigcup_{\substack{\mathbf{W} \subset \mathbf{V} \\ \Gamma_\mathbf{W} \text{ disconnected}}} H_\mathbf{W}.$$

It follows that the irreducible components of  $\mathcal{R}_1(G_\Gamma)$  are indexed by the subsets  $\mathbf{W} \subset \mathbf{V}$ , maximal among those for which  $\Gamma_\mathbf{W}$  is disconnected.

Using Theorems 2.1 and 3.1, we gave in [11] the following answer to Serre's problem for right-angled Artin groups.

**Theorem 3.2** ([11]). *Let  $\Gamma = (\mathbf{V}, \mathbf{E})$  be a finite simplicial graph, with associated right-angled Artin group  $G_\Gamma$ . The following are equivalent.*

- (i) *The group  $G_\Gamma$  is quasi-Kähler.*
- (ii) *The group  $G_\Gamma$  is quasi-projective.*
- (iii) *The group  $G_\Gamma$  satisfies the isotropicity condition from Theorem 2.1(1).*
- (iv) *The group  $G_\Gamma$  is a product of finitely generated free groups.*
- (v) *The graph  $\Gamma$  is a complete multipartite graph  $K_{n_1, \dots, n_r} = \overline{K}_{n_1} * \dots * \overline{K}_{n_r}$ .*

**3.2.** In order to apply Theorem 2.1 to the Bestvina-Brady groups, we need a description of their resonance varieties.

The inclusion  $\nu: N_\Gamma \rightarrow G_\Gamma$  induces a homomorphism  $\nu^*: H^1(G_\Gamma, \mathbb{C}) \rightarrow H^1(N_\Gamma, \mathbb{C})$ . Note that if  $|\mathbf{V}| = 1$ , then  $N_\Gamma = \{1\}$  and thus  $\mathcal{R}_1(N_\Gamma)$  is empty.

**Theorem 3.3** ([21]). *Let  $\Gamma$  be a graph, with connectivity  $\kappa(\Gamma)$ . Suppose the flag complex  $\Delta_\Gamma$  is simply-connected, and  $|\mathbf{V}| > 1$ .*

- (1) *If  $\kappa(\Gamma) = 1$ , then  $\mathcal{R}_1(N_\Gamma) = H^1(N_\Gamma, \mathbb{C})$ .*
- (2) *If  $\kappa(\Gamma) > 1$ , then the irreducible components of  $\mathcal{R}_1(N_\Gamma)$  are the subspaces  $H'_\mathbf{W} = \nu^*(H_\mathbf{W})$ , of dimension  $|\mathbf{W}|$ , one for each subset  $\mathbf{W} \subset \mathbf{V}$ , maximal among those for which  $\Gamma_\mathbf{W}$  is disconnected.*

**3.3.** To analyze the isotropicity properties of the resonance components in the above theorem, we need a precise description of the cohomology ring in degrees 1 and 2, for both right-angled Artin groups and Bestvina-Brady groups.

For  $G_\Gamma$ , the cohomology ring can be identified with the exterior Stanley-Reisner ring of the flag complex:  $H^*(G_\Gamma)$  is the quotient of the exterior algebra on generators  $v^*$  in degree 1, indexed by the vertices  $v \in \mathbf{V}$ , modulo the ideal generated by the monomials  $v^*w^*$  for which  $\{v, w\}$  is not an edge of  $\Gamma$ . It follows that an additive basis for  $H^k(G_\Gamma)$  is indexed by the complete  $k$ -subgraphs of  $\Gamma$ .

Viewing the homomorphism  $\nu: G_\Gamma \rightarrow \mathbb{Z} \hookrightarrow \mathbb{C}$  as an element in  $H^1(G_\Gamma, \mathbb{C})$ , we have the following.

**Theorem 3.4** ([21]). *Suppose  $\pi_1(\Delta_\Gamma) = 0$ . Then  $\iota^*: H^*(G_\Gamma, \mathbb{C}) \rightarrow H^*(N_\Gamma, \mathbb{C})$  induces a ring homomorphism  $\iota^*: H^*(G_\Gamma, \mathbb{C})/(\nu \cdot H^*(G_\Gamma, \mathbb{C})) \rightarrow H^*(N_\Gamma, \mathbb{C})$ , which is an isomorphism in degrees  $* \leq 2$ .*

In other words,  $\iota^*: H^1(G_\Gamma, \mathbb{C}) \rightarrow H^1(N_\Gamma, \mathbb{C})$  is onto, has one dimensional kernel generated by  $\nu$ , and fits into the following commuting diagram:

$$(1) \quad \begin{array}{ccc} \bigwedge^2 H^1(G_\Gamma, \mathbb{C}) & \xrightarrow{\wedge^2 \iota^*} & \bigwedge^2 H^1(N_\Gamma, \mathbb{C}) \\ \downarrow \cup_{G_\Gamma} & & \downarrow \cup_{N_\Gamma} \\ H^2(G_\Gamma, \mathbb{C}) & \longrightarrow & H^2(G_\Gamma, \mathbb{C})/\nu H^1(G_\Gamma, \mathbb{C}) \cong H^2(N_\Gamma, \mathbb{C}) \end{array}$$

#### 4. The quasi-Kähler condition for the groups $N_\Gamma$

In this section, we establish implication (Q1)  $\Rightarrow$  (Q3) from Theorem 1.1 in the Introduction: the implication follows at once from Lemmas 4.1 and 4.3 below. We conclude the section with proofs of Corollaries 1.2, 1.3, and 1.4.

**4.1.** Let  $\Gamma = (\mathbb{V}, \mathbb{E})$  be a finite simple graph. Denote by  $\mathbb{T}$  the set of triangles in  $\Gamma$ . Suppose the Bestvina-Brady group  $N_\Gamma$  is finitely presented. The following then hold:

**I.** The flag complex  $\Delta_\Gamma$  must be simply-connected, as shown in [3].

**II.** An explicit finite presentation for  $N_\Gamma$  is given by Dicks and Leary [9]. Fix a linear order on  $\mathbb{V}$ , and orient the edges increasingly. A triple of edges  $(e, f, g)$  forms a directed triangle if  $e = \{u, v\}$ ,  $f = \{v, w\}$ ,  $g = \{u, w\}$ , and  $u < v < w$ . Then:

$$(2) \quad N_\Gamma = \langle e \in \mathbb{E} \mid ef = fe, ef = g \text{ if } (e, f, g) \text{ is a directed triangle} \rangle.$$

**III.** The group  $N_\Gamma$  is 1-formal, as shown in Proposition 6.1 from [21].

**4.2.** Now suppose  $N_\Gamma = \pi_1(M)$ , with  $M$  a quasi-Kähler manifold. Then  $N_\Gamma$  is finitely presented, and thus 1-formal, by the above. Therefore, Theorem 2.1 applies to  $N_\Gamma$ .

**Lemma 4.1.** *Suppose  $N_\Gamma$  is a quasi-Kähler group. If there exists an edge  $e_0 \in \mathbb{E}$  not included in any triangle in  $\mathbb{T}$ , then  $\Gamma$  is a tree.*

*Proof.* We may safely assume  $\Gamma$  is not  $K_1$  or  $K_2$ , to avoid degenerate cases. Since  $\pi_1(\Delta_\Gamma) = 0$ , it is enough to show that  $\Gamma$  has no triangles. Indeed, if  $\mathbb{T} = \emptyset$ , then the 2-skeleton of  $\Delta_\Gamma$  coincides with  $\Gamma$ . Hence,  $\pi_1(\Gamma) = 0$ , and so  $\Gamma$  is a tree.

Write  $\mathbb{E}' = \mathbb{E} \setminus \{e_0\}$ . From the Dicks-Leary presentation (2), we find that  $N_\Gamma = \mathbb{Z} * N'$ , where  $\mathbb{Z}$  is the cyclic group generated by  $e_0$ , and

$$N' = \langle e \in \mathbb{E}' \mid ef = fe, ef = g \text{ if } (e, f, g) \text{ is a directed triangle} \rangle.$$

We may assume that  $b_1(N') \neq 0$ , for otherwise  $\Gamma = K_2$ . From [19, Corollary 5.4], we find that  $\mathcal{R}_1(N_\Gamma) = H^1(N_\Gamma, \mathbb{C})$ . Moreover, by [11, Lemma 7.4], the cup-product map  $\cup_{N_\Gamma}$  vanishes.

Suppose  $\tau = \{u, v, w\}$  is a triangle in  $\mathbb{T}$ . By the above,  $\iota^*(u^*) \cup_{N_\Gamma} \iota^*(v^*) = 0$  in  $H^2(N_\Gamma, \mathbb{C})$ . Write  $\nu = \sum_{z \in \mathbb{V}} z^* \in H^1(G_\Gamma, \mathbb{C})$ . From diagram (1), we find that  $u^* \cup_{G_\Gamma} v^* \cup_{G_\Gamma} \nu = 0$  in

$H^3(G_\Gamma, \mathbb{C})$ . Hence:

$$\sum_{z \in \mathbf{V}} u^* \cup_{G_\Gamma} v^* \cup_{G_\Gamma} z^* = 0.$$

But this sum contains the basis element  $u^*v^*w^*$  of  $H^3(G_\Gamma, \mathbb{C})$ , a contradiction.  $\square$

**Lemma 4.2.** *Suppose  $N_\Gamma$  is a quasi-Kähler group, and every edge of  $\Gamma$  belongs to a triangle. Then  $\kappa(\Gamma) > 1$ .*

*Proof.* Suppose  $\Gamma$  has connectivity 1. Then, by Theorem 3.3(1), we must have  $\mathcal{R}_1(N_\Gamma) = H^1(N_\Gamma, \mathbb{C})$ . Furthermore, this linear space must be either 0- or 1-isotropic, by Theorem 2.1(1). That is, the cup-product map  $\cup_{N_\Gamma}: \bigwedge^2 H^1(N_\Gamma, \mathbb{C}) \rightarrow H^2(N_\Gamma, \mathbb{C})$  either vanishes, or has 1-dimensional image and is non-degenerate.

On the other hand, we know from Theorem 3.4 that the map  $\cup_{N_\Gamma}$  is surjective. Thus, we must have  $b_2(N_\Gamma) \leq 1$ . Moreover, by Proposition 7.1 in [21], we have  $b_2(N_\Gamma) = |\mathbf{E}| - |\mathbf{V}| + 1$ , since  $\Delta_\Gamma$  is simply-connected. Thus,  $|\mathbf{V}| - 1 \leq |\mathbf{E}| \leq |\mathbf{V}|$ .

If  $|\mathbf{E}| = |\mathbf{V}| - 1$ , then  $\Gamma$  is a tree, contradicting the assumption that every edge belongs to a triangle. If  $|\mathbf{E}| = |\mathbf{V}|$ , then  $\Gamma$  is obtained from a tree by adding exactly one edge. Using again the assumption that every edge belongs to a triangle, it is readily seen that  $\Gamma = K_3$ . But  $\kappa(K_3) > 1$ , a contradiction.  $\square$

**Lemma 4.3.** *Suppose  $N_\Gamma$  is a quasi-Kähler group, and every edge of  $\Gamma$  belongs to a triangle. Then  $\Gamma = K_{n_1, \dots, n_r}$ , with either some  $n_i = 1$ , or all  $n_i \geq 2$  and  $r \geq 3$ .*

*Proof.* If  $\Gamma$  is a complete graph  $K_n = K_{1, \dots, 1}$ , there is nothing to prove. Otherwise, all irreducible components  $H'_W$  of  $\mathcal{R}_1(N_\Gamma)$  are positive-dimensional, by Lemma 4.2 and Theorem 3.3(2). Moreover, each subspace  $H'_W$  is 0- or 1-isotropic, by Theorem 2.1(1).

**Claim.** The resonance obstruction from Theorem 2.1(1) holds for  $G_\Gamma$ .

Assuming this claim, the proof is completed as follows. By Theorem 3.2, the graph  $\Gamma$  must be a complete multipartite graph  $K_{n_1, \dots, n_r}$ . If all  $n_i \geq 2$ , then necessarily  $r \geq 3$ , by the simply-connectivity assumption on  $\Delta_\Gamma$ , and we are done.

Let  $\bigcup_W H_W$  be the decomposition into irreducible components of  $\mathcal{R}_1(G_\Gamma)$  from Theorem 3.1, where  $W$  runs through the non-empty, maximal subsets of  $\mathbf{V}$  for which  $\Gamma_W$  is disconnected. We will prove that each induced subgraph  $\Gamma_W$  is a discrete graph. This will imply that each subspace  $H_W$  is a 0-isotropic subspace of  $H^1(G_\Gamma, \mathbb{C})$ , of dimension at least 2, thereby showing that  $G_\Gamma$  satisfies the resonance obstructions, as claimed.

Let  $\Gamma_W = \bigsqcup_j \Gamma_{W_j}$  be the decomposition into connected components of the disconnected graph  $\Gamma_W$ . Denote by  $m(W)$  the number of non-discrete connected components.

First suppose  $m(W) \geq 2$ . Then, there exist disjoint edges,  $e_1$  and  $e_2$ , in  $\Gamma_W$ . Recall that each edge  $e \in \mathbf{E}$  can be viewed as a basis element of  $H^2(G_\Gamma, \mathbb{C})$ . From the isotropicity of  $H'_W$ , we deduce that  $\iota^*(e_1)$  and  $\iota^*(e_2)$  are linearly dependent elements in  $H^2(N_\Gamma, \mathbb{C})$ . But this is impossible. Indeed, if  $\alpha_1 \iota^*(e_1) + \alpha_2 \iota^*(e_2) = 0$  is a linear dependence, then it follows from diagram (1) that  $(\alpha_1 e_1 + \alpha_2 e_2) \cdot \nu = 0$  in  $H^3(G_\Gamma, \mathbb{C})$ . Expanding the product, we find that

$$(3) \quad \alpha_1 \sum_{\tau \in \mathbf{T}: \tau \supset e_1} \tau + \alpha_2 \sum_{\tau \in \mathbf{T}: \tau \supset e_2} \tau = 0.$$

On one hand, we know that there are triangles  $\tau_1$  and  $\tau_2$  containing edges  $e_1$  and  $e_2$ , respectively. On the other hand, all basis elements  $\tau$  appearing in (3) are distinct, since  $e_1 \cap e_2 = \emptyset$ . Thus,  $\alpha_1 = \alpha_2 = 0$ .

Now suppose  $m(W) = 1$ . Write  $W = W' \sqcup W''$ , with  $\Gamma_{W'}$  discrete and non-empty,  $\Gamma_{W''}$  containing at least an edge, and with no edge joining  $W'$  to  $W''$ . We then have a non-trivial orthogonal decomposition with respect to  $\cup_{N_\Gamma}$ ,

$$(4) \quad H'_W = H'_{W'} \oplus H'_{W''}.$$

As above, we find that the restriction of  $\cup_{N_\Gamma}$  to  $\bigwedge^2 H'_{W''}$  is non-zero. Thus,  $H'_W$  cannot be 0-isotropic. On the other hand, any non-zero element in  $H'_{W'}$  is orthogonal to all of  $H'_W$ ; thus the restriction of  $\cup_{N_\Gamma}$  to  $H'_W$  is a degenerate form. Hence,  $H'_W$  cannot be 1-isotropic, either. This contradicts the isotropicity property of  $N_\Gamma$ .

Thus,  $m(W) = 0$ , and  $\Gamma_W$  is discrete, as asserted. This ends the proof.  $\square$

**4.3. Proof of Corollary 1.2.** The classification of quasi-Kähler Bestvina-Brady groups follows from Theorem 1.1. The passage from graphs to groups is achieved by using the following simple facts: if  $\Gamma$  is a tree on  $n$  vertices, then  $N_\Gamma = F_{n-1}$ ; if  $\Gamma = \Gamma_1 * \Gamma_2$ , then  $G_\Gamma = G_{\Gamma_1} \times G_{\Gamma_2}$ ; if  $\Gamma = K_1 * \Gamma'$ , then  $N_\Gamma = G_{\Gamma'}$ .

Next, note that the groups  $G$  in classes (1)–(3) have a finite  $K(G, 1)$ , while those in class (4) don't, by [3]. The groups in (1)–(3) are distinguished by their Poincaré polynomials. As for the groups in class (4), they are distinguished by their resonance varieties. Indeed, if  $\Gamma = K_{n_1, \dots, n_r}$ , with all  $n_i \geq 2$  and  $r \geq 3$ , then  $\kappa(\Gamma) > 1$  and  $\pi_1(\Delta_\Gamma) = 0$ ; thus, by Theorem 3.3(2), the variety  $\mathcal{R}_1(N_\Gamma)$  decomposes into  $r$  irreducible components, of dimensions  $n_1, \dots, n_r$ .

This finishes the proof of Corollary 1.2.  $\square$

**4.4. Proof of Corollary 1.3.** Implications (K3)  $\Rightarrow$  (K4) and (K2)  $\Rightarrow$  (K1) are clear. The fact that the torus  $T^{2n}$  is a smooth projective manifold proves (K4)  $\Rightarrow$  (K2).

We are left with proving (K1)  $\Rightarrow$  (K3). Suppose  $N_\Gamma$  is a Kähler group. Then of course  $N_\Gamma$  is quasi-Kähler, so  $\Gamma$  must be one of the graphs described in Theorem 1.1 (Q3).

Assume  $\Gamma$  is neither a complete graph, nor a complete multipartite graph  $K_{n_1, \dots, n_r}$  with all  $n_i \geq 2$  and  $r \geq 3$ . By the same argument as in the proof of Corollary 1.2, we infer that  $N_\Gamma$  must be of the form  $F_n \times N'$ , with  $n \geq 2$ . But this cannot be a Kähler group, by a result of Johnson and Rees, see [14, Theorem 3].

Suppose now  $\Gamma = K_{n_1, \dots, n_r}$ , with all  $n_i \geq 2$  and  $r \geq 3$ . It follows from Theorem 2.1(2) that all positive-dimensional irreducible components of  $\mathcal{R}_1(N_\Gamma)$  must be 1-isotropic. But this contradicts Theorem 3.3(2), which predicts only 0-isotropic components.

Finally, if  $\Gamma = K_n$ , note that  $b_1(N_\Gamma) = n - 1$ . Since the odd Betti numbers of a compact Kähler manifold are even,  $n$  must be odd.

This finishes the proof of Corollary 1.3.  $\square$

**4.5. Proof of Corollary 1.4.** Let  $G = N_\Gamma$  be a quasi-projective Bestvina-Brady group, as classified in Corollary 1.2.

If  $G$  is of type (1), (2), or (3), then  $G$  is the fundamental group of a space of the form  $M = \prod_{i=1}^q (\mathbb{C} \setminus \{n_i \text{ points}\})$ , an aspherical, smooth quasi-projective variety.

If  $G$  is of type (4), then  $G$  is not of type  $\text{FP}_\infty$ , by [3] (see also Proposition 7.1 and Remark 5.8 in [21] for a direct proof). As is well-known, a finite-index subgroup  $\pi'$  of a group  $\pi$  is of type  $\text{FP}_\infty$  if and only if  $\pi$  is of type  $\text{FP}_\infty$ ; see [6, Prop. VIII.5.1]. In particular, finite groups are of type  $\text{FP}_\infty$ . Moreover, if  $\pi$  is an extension of  $\pi'$  by a finite group, then  $\pi$  is of type  $\text{FP}_\infty$  if and only if  $\pi'$  is; see [4, Proposition 2.7].

Thus,  $G$  cannot be commensurable up to finite kernels to a group  $\pi$  of type  $\text{FP}_\infty$ , and so, a fortiori, to a group  $\pi$  admitting a finite-type  $K(\pi, 1)$ . We conclude that  $G$  is not commensurable up to finite kernels to any group of the form  $\pi = \pi_1(M)$ , where  $M$  is an aspherical, quasi-projective variety.

This finishes the proof of Corollary 1.4.  $\square$

**Remark 4.4.** Two groups,  $G$  and  $\pi$ , are *elementarily commensurable up to finite kernels* if there is a homomorphism  $\varphi: G \rightarrow \pi$  with finite kernel and with finite-index image; the associated equivalence relation is the one defined at the end of §1.1. A stronger equivalence relation is defined as follows:  $G$  and  $\pi$  are *commensurable* if there exist finite-index subgroups,  $G' < G$  and  $\pi' < \pi$ , such that  $G' \cong \pi'$ .

Being elementarily commensurable up to finite kernels is not a symmetric relation. An example, due to Milnor, can be found within the class of finitely generated, nilpotent, torsion-free groups; see [13, p. 25]. It is worth noting that, in this class of groups, the two commensurability relations coincide: they both amount to saying that  $G$  and  $\pi$  have the same rationalization; see [13, Theorem I.3.3] and [2, Lemma 2.8 on p. 19].

## 5. Realizability by quasi-projective varieties

In this section, we prove implication (Q3)  $\Rightarrow$  (Q2) from Theorem 1.1. The implication is easily proved for groups belonging to the classes (1)–(3) listed in Corollary 1.2, by taking products of punctured projective lines. To realize the groups in class (4) by smooth quasi-projective varieties, we will appeal to results on the topology of fibers of polynomial mappings, see [5, 10].

Let  $\mathcal{A} = \{H_1, \dots, H_d\}$  be an affine essential hyperplane arrangement in  $\mathbb{C}^n$ . We set  $M = \mathbb{C}^n \setminus X$ , with  $X$  being the union of all the hyperplanes in  $\mathcal{A}$ . Consider a  $d$ -tuple of positive integers,  $e = (e_1, \dots, e_d)$ , with  $\gcd(e_1, \dots, e_d) = 1$ . For any  $i = 1, \dots, d$ , let  $\ell_i = 0$  be an equation for the hyperplane  $H_i$  and consider the product

$$f_e = \prod_{i=1}^d \ell_i^{e_i} \in \mathbb{C}[x_1, \dots, x_n].$$

Let  $d_e = e_1 + \dots + e_d$  be the degree of the polynomial  $f_e$ . Note that  $f_e = 0$  is a (possibly non-reduced) equation for the union  $X$ . The induced morphism

$$\nu_e = (f_e)_\# : \pi_1(M) \rightarrow \pi_1(\mathbb{C}^*) = \mathbb{Z}$$

sends an elementary (oriented) loop about the hyperplane  $H_j$  to  $e_j \in \mathbb{Z}$ .

When the arrangement  $\mathcal{A}$  is central, i.e.,  $0 \in H_i$  for all  $i = 1, \dots, d$ , the above polynomial  $f_e$  is homogeneous, and there is a lot of interest in the associated Milnor fiber  $F_e = f_e^{-1}(1)$ . In

particular, we have the following exact sequence of groups:

$$1 \longrightarrow \pi_1(F_e) \longrightarrow \pi_1(M) \longrightarrow \pi_1(\mathbb{C}^*) = \mathbb{Z} \longrightarrow 0 .$$

The main result of this section is the following.

**Theorem 5.1.** *For any affine essential hyperplane arrangement in  $\mathbb{C}^n$  with  $n \geq 3$ , any relatively prime integers  $e$  as above and any  $t \in \mathbb{C}^*$ , there is an exact sequence of groups*

$$1 \longrightarrow \pi_1(F_{e,t}) \longrightarrow \pi_1(M) \xrightarrow{\nu_e} \pi_1(\mathbb{C}^*) = \mathbb{Z} \longrightarrow 0 ,$$

where  $F_{e,t} = f_e^{-1}(t)$ .

*Proof.* It is known that the mapping  $f_e: M \rightarrow \mathbb{C}^*$  has only finitely many singular points, see [10, Corollary 2.2]. Let  $R > 0$  be a real number such that the disc

$$D_R = \{z \in \mathbb{C} ; |z| \leq R\}$$

contains in its interior all the critical values of  $f_e$ . We set  $S_R^1 = \{z \in \mathbb{C} ; |z| = R\}$ . Let  $M_R = M \cap f_e^{-1}(D_R)$  and  $E_R = f_e^{-1}(S_R^1)$ . The main result in [10], Theorem 2.1(ii), says roughly that our polynomial mapping  $f_e$  behaves like a proper mapping. This comes from the fact that all the fibers of  $f_e$  are transversal to large enough spheres centered at the origin of  $\mathbb{C}^n$ , similarly to the case of tame polynomials considered in [5]. In particular, this shows that the bifurcation set of the mapping  $f_e: \mathbb{C}^n \rightarrow \mathbb{C}$ , i.e., the minimal set to be deleted in order to have a locally trivial fibration, coincides with the set of critical values of  $f_e$ . Specifically, the following properties hold.

(A) The restriction of  $f_e$  induces a locally trivial fibration  $f_e: E_R \rightarrow S_R^1$ . In particular, we get an exact sequence

$$1 \longrightarrow \pi_1(F_{e,R}) \longrightarrow \pi_1(E_R) \longrightarrow \pi_1(S_R^1) = \mathbb{Z} \longrightarrow 0 .$$

(B) The inclusion  $M_R \hookrightarrow M$  is a homotopy equivalence.

(C) The space  $M_R$  has the homotopy type of a CW-complex obtained from  $E_R$  by attaching finitely many  $n$ -cells. This is a general trick in this situation, see [5]. In particular, the inclusion  $E_R \hookrightarrow M$  induces an isomorphism at the level of fundamental groups if  $n \geq 3$ .

(D) Any smooth fiber  $F_{e,t} = f_e^{-1}(t)$  of  $f_e$  is diffeomorphic to the fiber  $F_{e,R} = f_e^{-1}(R)$ . And any singular fiber  $F_{e,t} = f_e^{-1}(t)$  of  $f_e$  for  $t \neq 0$  is obtained from the fiber  $F_{e,R} = f_e^{-1}(R)$  by adding a finite number of  $n$ -cells. In particular, if  $n \geq 3$ , there are natural isomorphisms  $\pi_1(F_{e,t}) \rightarrow \pi_1(F_{e,R})$  for any  $t \in \mathbb{C}^*$ .

These properties allow us to transform the exact sequence in (A) to get our claimed exact sequence.  $\square$

The quasi-projectivity of the groups  $N_{K_{n_1, \dots, n_r}}$  ( $r \geq 3$ ) follows via a simple remark: groups of the form  $F_{n_1} \times \dots \times F_{n_r}$  can be realized by affine arrangements; see the proof of the Corollary below. Related results, on the topology of *infinite* cyclic covers for arrangements having fundamental groups of this form, may be found in [7].

**Corollary 5.2.** *The Bestvina-Brady groups associated to the graphs  $K_{n_1, \dots, n_r}$  with  $r \geq 3$  are quasi-projective.*

*Proof.* Let  $\mathcal{A}$  be the arrangement in  $\mathbb{C}^r$  defined by the polynomial

$$f = (x_1 - 1) \cdots (x_1 - n_1) \cdots (x_r - 1) \cdots (x_r - n_r).$$

Theorem 5.1 may be applied to  $\mathcal{A}$  and  $e = (1, \dots, 1)$ . Clearly,  $\pi_1(M)$  may be identified with the right-angled Artin group  $G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$  defined by the graph  $\Gamma = K_{n_1, \dots, n_r}$ , in such a way that  $\nu_e$  becomes identified with the length homomorphism  $\nu: G_\Gamma \rightarrow \mathbb{Z}$ . Hence,  $N_\Gamma$  is isomorphic to the fundamental group of the smooth fiber  $F_{e,t} = f_e^{-1}(t)$  of  $f_e$ .  $\square$

This Corollary completes the proof of Theorem 1.1.

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