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ALGEBRAIC INVARIANTS FOR BESTVINA-BRADY GROUPS

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Abstract

Bestvina-Brady groups arise as kernels of length homomorphisms $G_\Gamma \to \mathbb{Z}$ from right-angled Artin groups to the integers. Under some connectivity assumptions on the flag complex Δ_Γ, we compute several algebraic invariants of such a group N_Γ, directly from the underlying graph Γ. As an application, we give examples of finitely presented Bestvina-Brady groups which are not isomorphic to any Artin group or arrangement group.

1. *Introduction and statement of results*

1.1. *Bestvina-Brady groups*

Given a finite simple graph $\Gamma = (\mathsf{V}, \mathsf{E})$, the corresponding right-angled Artin group G_Γ has presentation with a generator v for each vertex $v \in \mathsf{V}$, and a commutator relation $vw = wv$ for each edge $\{v, w\} \in \mathsf{E}$. The Bestvina-Brady group (or, Artin kernel) associated to Γ, denoted N_Γ, is the kernel of the "length" homomorphism to the additive group of integers, $\nu\colon G_\Gamma \to \mathbb{Z}$, which sends each generator $v \in \mathsf{V}$ to 1.

As shown by Bestvina and Brady in their seminal paper [**1**], the geometric and homological finiteness properties of the group N_Γ are intimately connected to the topology of the flag complex Δ_Γ. For example, N_Γ is finitely generated if and only if the graph Γ is connected; and N_Γ is finitely presented if and only if Δ_Γ is simply-connected. The groups N_Γ are complicated enough that a counterexample to either the Eilenberg-Ganea conjecture or the Whitehead asphericity conjecture can be constructed from them.

It is known that two right-angled Artin groups G_Γ and $G_{\Gamma'}$ are isomorphic if and only if the corresponding graphs, Γ and Γ', are isomorphic; see [**15**], [**7**]. No such simple classification of the Bestvina-Brady groups is possible. Indeed, if Γ is a tree on n vertices, then $N_\Gamma = F_{n-1}$ (the free group of rank $n-1$), as follows from [**4**]. Thus, for any $n \geq 4$, there exist graphs Γ and Γ' on n vertices such that $\Gamma \ncong \Gamma'$, yet $N_\Gamma \cong N_{\Gamma'}$.

We study here a variety of algebraic invariants of a group N_Γ (mainly derived from the lower central series and the cohomology ring), showing how to compute these invariants directly from the graph Γ, provided some connectivity assumptions

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on Δ_Γ are satisfied. In turn, such invariants can be used to distinguish Bestvina-Brady groups, both among themselves, and from other, related classes of groups, such as Artin groups, or arrangement groups.

1.2. *LCS quotients and Chen groups*

We start by studying invariants derived from the lower central series. For a group G, this series is defined by $\gamma_1 G = G$ and $\gamma_{k+1} G = (\gamma_k G, G)$, where $(x, y) = xyx^{-1}y^{-1}$. The direct sum of the successive quotients, $\operatorname{gr}(G) = \bigoplus_{k\geq 1} \gamma_k G/\gamma_{k+1} G$, is the *associated graded Lie algebra* of G. The Lie bracket, induced from the group commutator, is compatible with the grading. By construction, the Lie algebra $\operatorname{gr}(G)$ is generated by $\operatorname{gr}_1(G)$. Consequently, the derived Lie subalgebra, $\operatorname{gr}'(G)$, coincides with $\bigoplus_{k\geq 2} \operatorname{gr}_k(G)$.

THEOREM 1.1. *Let $\Gamma = (\mathsf{V}, \mathsf{E})$ be a connected graph, and let N_Γ be the corresponding Bestvina-Brady group. The associated graded Lie algebra $\operatorname{gr}(N_\Gamma)$ is torsion-free, with graded ranks $\phi_k = \operatorname{rank} \operatorname{gr}_k(N_\Gamma)$ given by*

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = \frac{P_\Gamma(-t)}{1 - t},$$

where $P_\Gamma(t) = \sum_{k\geq 0} f_k(\Gamma) t^k$ is the clique polynomial of Γ, with $f_k(\Gamma)$ equal to the number of k-cliques of Γ. Moreover, $\operatorname{gr}'(N_\Gamma)$ is isomorphic (as a graded Lie algebra) to the derived Lie algebra of $\mathfrak{H}_\Gamma = \operatorname{Lie}(\mathsf{V})/([v, w] = 0 \text{ if } \{v, w\} \in \mathsf{E})$.

For a group G, let $G' = \gamma_1 G$ be the derived group, and $G'' = (G')'$ the second derived group. Note that $H_1(G) = G/G'$ is the maximal abelian quotient of G, whereas G/G'' is the maximal metabelian quotient. Define the *Chen Lie algebra* of G to be $\operatorname{gr}(G/G'')$.

THEOREM 1.2. *Let $\Gamma = (\mathsf{V}, \mathsf{E})$ be a connected graph, and let N_Γ be the corresponding Bestvina-Brady group. The Chen Lie algebra $\operatorname{gr}(N_\Gamma/N_\Gamma'')$ is torsion-free, with graded ranks θ_k given by $\theta_1 = |\mathsf{V}| - 1$ and*

$$\sum_{k=2}^{\infty} \theta_k t^k = Q_\Gamma\Big(\frac{t}{1 - t}\Big),$$

where $Q_\Gamma(t) = \sum_{j\geq 2} \Big(\sum_{\mathsf{W}\subset \mathsf{V}\colon |\mathsf{W}|=j} \tilde{b}_0(\Gamma_{\mathsf{W}})\Big) t^j$ is the cut polynomial of Γ. Moreover, $\operatorname{gr}'(N_\Gamma/N_\Gamma'')$ is isomorphic to the derived Lie algebra of $\mathfrak{H}_\Gamma/\mathfrak{H}_\Gamma''$.

These two theorems rely on the analogous computations for right-angled Artin groups, done in [**22**]. The proofs involve a homological analysis of the extension $1 \to N_\Gamma \to G_\Gamma \to \mathbb{Z} \to 0$, based on the Salvetti complex for G_Γ. This analysis shows that $\mathbb{Z}$ acts trivially on $H_1(N_\Gamma)$, thereby allowing us to invoke the Falk-Randell lemma [**11**].

1.3. *Cohomology ring and formality*

Next, we turn to cohomological invariants. If G is a group, with Eilenberg-MacLane space $K(G, 1)$, then the cohomology of G with coefficients in a commutative ring R is defined as $H^*(G, R) := H^*(K(G, 1), R)$, with ring structure given by the cup product. The group G is said to be 1-formal if its Malcev Lie algebra is quadratically presented, cf. [**24**]. In this case, the rational associated graded Lie algebra $\operatorname{gr}(G)\otimes\mathbb{Q}$ is isomorphic to the rational holonomy Lie algebra, $\mathfrak{H}_{\mathbb{Q}}(G)$, which in turn is determined by the cohomology ring in low degrees, $H^{\leq 2}(G,\mathbb{Q})$.

For a right-angled Artin group G_Γ, the cohomology ring can be identified with the exterior Stanley-Reisner ring of the flag complex: $H^*(G_\Gamma)$ is the quotient of the exterior algebra on generators v^* in degree 1, indexed by the vertices $v \in \mathsf{V}$, modulo the ideal generated by the monomials v^*w^* for which $\{v,w\}$ is not an edge of Γ; see [**14**]. Furthermore, the group G_Γ is 1-formal; see [**13**].

Denote by $\iota\colon N_\Gamma \to G_\Gamma$ the inclusion map of the kernel, and view the homomorphism $\nu\colon G_\Gamma \to \mathbb{Z} \hookrightarrow \mathbb{Q}$ as an element in $H^1(G_\Gamma,\mathbb{Q})$. The next theorem determines the rational cohomology ring of N_Γ, in low degrees. A more general result has been independently obtained by Leary and Saadetoğlu [**16**].

Theorem 1.3. *Suppose $\pi_1(\Delta_\Gamma)=0$. Then $\iota^*\colon H^*(G_\Gamma,\mathbb{Q}) \to H^*(N_\Gamma,\mathbb{Q})$ induces a ring homomorphism $\iota^*\colon H^*(G_\Gamma,\mathbb{Q})/(\nu\cdot H^*(G_\Gamma,\mathbb{Q})) \to H^*(N_\Gamma,\mathbb{Q})$, which is an isomorphism in degrees $*\leq 2$.*

When $\pi_1(\Delta_\Gamma)=0$, an explicit finite presentation for N_Γ was given by Dicks and Leary [**4**]. We use this presentation to show that the Bestvina-Brady group N_Γ is 1-formal.

1.4. *Cohomology jumping loci*

Let G be a finitely presented group, with character torus $\mathbb{T}_G = \operatorname{Hom}(G,\mathbb{C}^*)$. Identifying the point $\rho \in \mathbb{T}_G$ with a rank one local system ${}_\rho\mathbb{C}$ on an Eilenberg-MacLane space $K(G,1)$, we may define

$$\mathcal{V}_1(G) = \{\rho \in \mathbb{T}_G \mid H^1(G, {}_\rho\mathbb{C}) \neq 0\}.$$

The set $\mathcal{V}_1(G)$ is an algebraic subvariety of $\mathbb{T}_G$, called the *(first) characteristic variety* of G. Away from the origin, this variety coincides with the zero set of the annihilator of the Alexander invariant, $B(G)\otimes\mathbb{C}$.

Denote by $A = H^*(G,\mathbb{C})$ the cohomology algebra of G. For each $a \in A^1$, we have $a^2=0$, and so right-multiplication by a defines a cochain complex (A,a): $A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2$. Let $\mathcal{R}_1(G)$ be the set of points $a \in A^1$ where this complex fails to be exact,

$$\mathcal{R}_1(G) = \{a \in A^1 \mid H^1(A,a) \neq 0\}.$$

The set $\mathcal{R}_1(G)$ is a homogeneous algebraic variety in the affine space $A^1 = H^1(G,\mathbb{C})$, called the *(first) resonance variety* of G. Away from the origin, this variety coincides with the zero set of the annihilator of the infinitesimal Alexander invariant, $\mathfrak{B}(G)\otimes\mathbb{C}$.

In previous work [**22**], [**5**], we determined the resonance and characteristic varieties of right-angled Artin groups. Here, we determine these varieties for the finitely presented Bestvina-Brady groups.

Let $\mathbb{T}_{\mathsf{V}} = (\mathbb{C}^*)^{\mathsf{V}}$ be the character torus of G_Γ (of dimension $|\mathsf{V}|$). For a subset $\mathsf{W} \subset \mathsf{V}$, let $\mathbb{T}_{\mathsf{W}}$ be the coordinate subtorus supported on W. Similarly, let $H_{\mathsf{V}} = \mathbb{C}^{\mathsf{V}}$ be the Lie algebra of $\mathbb{T}_{\mathsf{V}}$, identified with $A^1 = H^1(G_\Gamma,\mathbb{C})$, and let H_{W} be the coordinate subspace supported on W. The inclusion $\iota\colon N_\Gamma \to G_\Gamma$ induces homomorphisms

$\iota^*\colon \mathrm{Hom}(G_\Gamma, \mathbb{C}^*) \to \mathrm{Hom}(N_\Gamma, \mathbb{C}^*)$ and $\iota^*\colon H^1(G_\Gamma, \mathbb{C}) \to H^1(N_\Gamma, \mathbb{C})$. Note that if $|\mathsf{V}| = 1$, then $N_\Gamma = \{1\}$ and thus both $\mathcal{V}_1(N_\Gamma)$ and $\mathcal{R}_1(N_\Gamma)$ are empty.

For a graph Γ on vertex set V, define the connectivity $\kappa(\Gamma)$ to be the maximum integer r so that, for any set of vertices W of size less than r, the full subgraph of Γ on vertex set $\mathsf{V} \setminus \mathsf{W}$ is connected.

THEOREM 1.4. *Let Γ be a graph. Suppose $\pi_1(\Delta_\Gamma) = 0$ and $|\mathsf{V}| > 1$.*

(i) *If $\kappa(\Gamma) = 1$, then $\mathcal{V}_1(N_\Gamma) = \mathrm{Hom}(N_\Gamma, \mathbb{C}^*)$ and $\mathcal{R}_1(N_\Gamma) = H^1(N_\Gamma, \mathbb{C})$.*

(ii) *If $\kappa(\Gamma) > 1$, then the irreducible components of $\mathcal{V}_1(N_\Gamma)$, respectively $\mathcal{R}_1(N_\Gamma)$, are the subtori $\mathbb{T}'_{\mathsf{W}} = \iota^*(\mathbb{T}_{\mathsf{W}})$, respectively the subspaces $H'_{\mathsf{W}} = \iota^*(H_{\mathsf{W}})$, of dimension $|\mathsf{W}|$, one for each subset $\mathsf{W} \subset \mathsf{V}$, maximal among those for which the induced subgraph Γ_{W} is disconnected.*

1.5. *Comparison with other classes of groups*

It turns out that Bestvina-Brady groups share many common features with other, much-studied classes of groups: finite-type Artin groups and fundamental groups of complements of complex hyperplane arrangements. We catalogue here some of these common features, and indicate certain overlaps between the various classes. As a counterpoint, and as an application of our methods, we give examples of finitely presented Bestvina-Brady groups which are not isomorphic to any group from those two other classes.

THEOREM 1.5. *There exists an infinite family of graphs $\{\Gamma_i\}_{i\in\mathbb{N}}$ such that the Bestvina-Brady group N_{Γ_i} is finitely presented, yet not isomorphic to either an Artin group, or an arrangement group.*

These graphs are obtained as the 1-skeleta of certain 'extra-special' triangulations of the 2-disk.

Using some of the machinery developed here, a complete classification of the Bestvina-Brady groups which can be realized as fundamental groups of quasi-projective varieties is given in [**6**].

1.6. *Organization of the paper*

We start in Section 2 with a review of the Bestvina-Brady groups, and a discussion of the Dicks-Leary presentation.

In Section 3 we recall the Salvetti complex for G_Γ, and use it to analyze the induced homomorphism $\iota_*\colon H_1(N_\Gamma) \to H_1(G_\Gamma)$.

In Section 4 we give presentations for the Alexander invariants $B(G_\Gamma)$ and $B(N_\Gamma)$.

In Section 5 we relate the graded Lie algebras attached to G_Γ and N_Γ, and prove Theorems 1.1 and 1.2.

In Section 6 we show that finitely presented Bestvina-Brady groups are 1-formal.

In Section 7 we compute the homology groups $H_*(N_\Gamma, \Bbbk)$, and prove Theorem 1.3.

In Section 8 we relate the characteristic and resonance varieties of G_Γ to those of N_Γ, and prove Theorem 1.4.

Finally, in Section 9, we compare finitely presented Bestvina-Brady groups with Artin groups and arrangement groups, and prove Theorem 1.5.

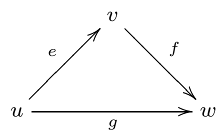

FIGURE 1. *A directed triangle*

2. *Bestvina-Brady groups*

Let Γ be a finite graph without loops or multiple edges, with vertex set V and edge set $\mathsf{E} \subset \binom{\mathsf{V}}{2}$. The flag complex of Γ, denoted Δ_Γ, is the maximal simplicial complex with 1-skeleton equal to Γ: the k-simplices of Δ_Γ correspond to the $(k+1)$-cliques of Γ.

To the graph Γ, there is associated a *right-angled Artin group*, G_Γ, with a generator v for each vertex in V, and with a commutator relation for each edge in E:

$$G_\Gamma = \langle v \in \mathsf{V} \mid vw = wv \ \text{ if } \ \{v, w\} \in \mathsf{E} \rangle. \tag{2.1}$$

For example, if Γ is the empty (or null) graph on n vertices, then $G_\Gamma = F_n$ (the free group of rank n), whereas if Γ is the complete graph K_n, then $G_\Gamma = \mathbb{Z}^n$.

DEFINITION 2.1. The *Bestvina-Brady group* associated to the graph $\Gamma = (\mathsf{V}, \mathsf{E})$, denoted N_Γ, is the kernel of the "length" homomorphism $\nu\colon G_\Gamma \to \mathbb{Z}$ which sends each generator $v \in \mathsf{V}$ to 1.

If $\iota\colon N_\Gamma \to G_\Gamma$ denotes the inclusion map, we have an exact sequence

$$1 \longrightarrow N_\Gamma \xrightarrow{\ \iota\ } G_\Gamma \xrightarrow{\ \nu\ } \mathbb{Z} \longrightarrow 0\,. \tag{2.2}$$

The group N_Γ need not be finitely generated. For example, if Γ is the empty graph on $n > 1$ vertices, then $N_\Gamma = \ker(\nu\colon F_n \twoheadrightarrow \mathbb{Z})$ is a free group of countably infinite rank. More generally, it was shown by Meier–VanWyk [**18**] and Bestvina–Brady [**1**] that the group N_Γ is finitely generated if and only if the graph Γ is connected.

Even if the graph Γ is connected, the group N_Γ may not have a finite presentation. For example, if Γ is a 4-cycle, then $G_\Gamma = F_2 \times F_2$; as noted by Stallings [**26**], $H_2(N_\Gamma)$ is not finitely generated, and so N_Γ is not finitely presented. Much more generally, Bestvina and Brady [**1**] showed that N_Γ is finitely presented if and only if the flag complex Δ_Γ is simply-connected. In this case, an explicit finite presentation was given by Dicks and Leary [**4**].

Fix a linear order on the vertices, and orient the edges increasingly. A triple of edges (e, f, g) forms a directed triangle if $e = \{u, v\}$, $f = \{v, w\}$, $g = \{u, w\}$, and $u < v < w$; see Figure 1.

THEOREM 2.2 (Dicks–Leary [**4**]). *Suppose the flag complex Δ_Γ is simply connected. Then N_Γ has presentation*

$$N_\Gamma = \langle e \in \mathsf{E} \mid ef = fe,\ ef = g \ \text{ if } (e, f, g) \text{ is a directed triangle} \rangle. \tag{2.3}$$

Moreover, the inclusion $\iota\colon N_\Gamma \to G_\Gamma$ is given by $\iota(e) = uv^{-1}$, for $e = \{u, v\}$ as above.

The Dicks-Leary presentation is far from being minimal (unless Γ is a tree). Indeed, there are $|\mathsf{E}|$ generators in (2.3), whereas $H_1(N_\Gamma)$ has rank $|\mathsf{V}|-1$, as we shall see in Proposition 4.2. Nevertheless, (2.3) can be simplified via Tietze moves to a presentation where all the relations are commutators.

Corollary 2.3. *If $\pi_1(\Delta_\Gamma)=0$, then N_Γ admits a commutator-relators presentation, $N_\Gamma = F/R$, with F the free group generated by the edges in a maximal tree T, and R a finitely generated normal subgroup of F'.*

Proof. Fix a maximal tree T for Γ. Suppose $e=\{u,v\}$ is an edge not in T. Picking a path $e_1,\dots,e_r$ in T connecting u to v, we see that $\iota(e)=\iota(e_1^{\epsilon_1}\dots e_r^{\epsilon_r})$ in G_Γ, for some suitable signs ϵ_i. Thus, $e=e_1^{\epsilon_1}\dots e_r^{\epsilon_r}$ in N_Γ. This shows that N_Γ is generated by the edges of T. Now note that N_Γ/N_Γ' is free abelian, of rank equal to the number of edges in T. Eliminating the redundant generators from (2.3), we arrive at the desired presentation. □

In certain situations, the Dicks-Leary presentation permits us to identify the group N_Γ in terms of better known groups.

Example 2.4. Suppose Γ is a tree on n vertices. Then Γ has no triangles, and $\pi_1(\Delta_\Gamma)=0$. Since Γ has $n-1$ edges, we see that $N_\Gamma=F_{n-1}$.

Example 2.5. Suppose Γ is the cone on Γ'. Then $G_\Gamma = G_{\Gamma'}\times\mathbb{Z}$, and so $N_\Gamma = G_{\Gamma'}$. In particular, if $\Gamma=K_n$, then $N_\Gamma=\mathbb{Z}^{n-1}$.

In general, though, N_Γ is not isomorphic to any right-angled Artin group, as we shall show later.

Noteworthy is the situation when Δ_Γ is a triangulation of the 2-disk. In this case, N_Γ admits a 2-dimensional $K(N_\Gamma,1)$, see [**3**, Corollary 2.3].

Definition 2.6. A triangulation of the disk is said to be *special* if it is obtained from a triangle by adding one triangle at a time, along a unique boundary edge.

Lemma 2.7. *Let Δ be a special triangulation of D^2, with* 1*-skeleton* $\Gamma=(\mathsf{V},\mathsf{E})$. *Then:*

(i) $2\,|\mathsf{V}|-|\mathsf{E}|=3$.
(ii) $\Delta_\Gamma=\Delta$.
(iii) *N_Γ admits a presentation with $|\mathsf{V}|-1$ generators and $|\mathsf{V}|-2$ commutator relators.*

Proof. By induction on the number t of triangles. Evidently, all statements hold for a single triangle. Now suppose Δ is a special triangulation with t triangles, and a directed triangle (e,f,g) is added along edge e, to form Δ'. In the process, one vertex and two edges are added, and so the quantity $2\,|\mathsf{V}|-|\mathsf{E}|$ does not change. The only new 3-cycle in the graph Γ' is the boundary of (e,f,g); thus, Δ' is a flag complex. Furthermore, if T is a maximal tree for Γ, we can build a maximal tree T' for Γ' by adding a new edge, say, f. The Dicks-Leary relation $ef=g$ (with e expressed as a word in the edges of T) may be used to eliminate the generator g. Thus, $N_{\Gamma'}$ has only one new generator, f, and only one new relation, $ef=fe$. □

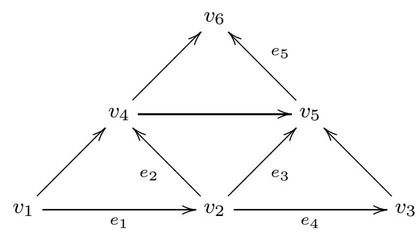

FIGURE 2. *A special triangulation of the 2-disk*

EXAMPLE 2.8. Let Γ be the graph in Figure 2. Choosing a maximal tree $\mathsf{T} = \{e_1, \dots, e_5\}$ as indicated, the presentation from Lemma 2.7(iii) reads as follows:

$$N_\Gamma = \langle e_1, \dots, e_5 \mid (e_1, e_2),\ (e_2, e_3),\ (e_3, e_4),\ e_5 e_2^{-1} e_3 = e_2^{-1} e_3 e_5 \rangle.$$

We shall see in Proposition 9.4 that $N_\Gamma \ncong G_{\Gamma'}$, no matter what the graph Γ' is.

3. *The Salvetti complex*

For a simple graph $\Gamma = (\mathsf{V}, \mathsf{E})$, let K_Γ be the CW-complex obtained by joining tori in the manner prescribed by the flag complex Δ_Γ. More precisely, if $T^n = (S^1)^{\times n}$ is the torus of dimension $n = |\mathsf{V}|$, with the usual CW-decomposition, then K_Γ is the subcomplex obtained by deleting the cells corresponding to the non-faces of Δ_Γ.

Clearly, the fundamental group of K_Γ is the right-angled Artin group G_Γ. In fact, K_Γ is an Eilenberg-MacLane space of type $K(G_\Gamma, 1)$; see [**18**]. Note that $H_k(K_\Gamma)$ is free abelian, of rank equal to the number of k-cliques in Γ; thus, the Poincaré polynomial of K_Γ equals the clique polynomial of the graph, $P_\Gamma(t)$.

We use the cell structure of the space K_Γ to describe a finite, free resolution of $\mathbb{Z}$, viewed as a trivial module over the group ring $\mathbb{Z}G_\Gamma$. This resolution was first determined by Salvetti [**25**], in the more general context of Artin groups, and further extended by Charney and Davis [**2**]. For the benefit of the reader, we include a self-contained, direct computation of the Salvetti complex in our particular case.

3.1. *A free $\mathbb{Z}G_\Gamma$-resolution of $\mathbb{Z}$*

For each subset $\mathsf{W} \subset \mathsf{V}$, let Γ_W be the induced subgraph of Γ on vertex set W. Let $G_\mathsf{W} = G_{\Gamma_\mathsf{W}}$ be the corresponding right-angled Artin group, and let $K_\mathsf{W} = K_{\Gamma_\mathsf{W}}$ be the corresponding CW-complex. The inclusion $\mathsf{W} \subset \mathsf{V}$ gives rise to a cellular inclusion map $j_\mathsf{W}\colon K_\mathsf{W} \to K_\Gamma$. The induced homomorphism, $(j_\mathsf{W})_\#\colon G_\mathsf{W} \to G_\Gamma$, is a split injection, with retract $G_\Gamma \to G_\mathsf{W}$ given on generators by $v \mapsto v$ if $v \in \mathsf{W}$, and $v \mapsto 1$ otherwise.

Denote by $C_\bullet = C_\bullet(K_\Gamma)$ the cellular chain complex of K_Γ. A basis for C_k is given by the complete k-subgraphs of Γ: to each complete subgraph on vertex set $\mathsf{W} \subset \mathsf{V}$, there corresponds a cell c_W. Since $C_\bullet$ is a sub-complex of the cellular chain complex of T^n, the boundary maps $C_k \to C_{k-1}$ are the zero maps.

Now let $\widetilde{C}_\bullet = (C_\bullet(\widetilde{K}_\Gamma), \partial_\bullet)$ be the equivariant chain complex of the universal cover of K_Γ. The augmentation map, $\epsilon\colon \widetilde{C}_0 \to \mathbb{Z}$, extends to a finite, free resolution $\widetilde{C}_\bullet \to \mathbb{Z}$ over the group ring $\mathbb{Z}G_\Gamma$.

Proposition 3.1. *Under the identification $\widetilde{C}_k = \mathbb{Z}G_\Gamma \otimes C_k$, the boundary map $\partial_k\colon \widetilde{C}_k \to \widetilde{C}_{k-1}$ is given by:*

$$\partial_k(c_{\mathsf{W}}) = \sum_{r=1}^{k} (-1)^{r-1}(v_{i_r} - 1)c_{\mathsf{W}\setminus\{i_r\}} \tag{3.1}$$

where $\mathsf{W} = \{v_{i_1}, \dots, v_{i_k}\}$ is a k-clique in Γ.

Proof. Let $K_{\mathsf{W}} = T^k$ be the corresponding CW-subcomplex of K_Γ. The equivariant chain complex $\widetilde{C}^{\mathsf{W}}_\bullet = (C_\bullet(\widetilde{K}_{\mathsf{W}}), \partial^{\mathsf{W}}_\bullet)$ is simply the Koszul complex on the variables from W. In particular, $\partial^{\mathsf{W}}_k(c_{\mathsf{W}})$ is given by the right-hand side of (3.1). The commutativity of the diagram

$$\begin{tikzcd} \mathbb{Z}G_{\mathsf{W}} \otimes C_k(K_{\mathsf{W}}) \arrow[r, "\partial^{\mathsf{W}}_k"] \arrow[d, "j_\# \otimes j_*"] & \mathbb{Z}G_{\mathsf{W}} \otimes C_{k-1}(K_{\mathsf{W}}) \arrow[d, "j_\# \otimes j_*"] \\ \mathbb{Z}G_\Gamma \otimes C_k(K_\Gamma) \arrow[r, "\partial_k"] & \mathbb{Z}G_\Gamma \otimes C_{k-1}(K_\Gamma) \end{tikzcd}$$

completes the proof. ∎

3.2. *An injectivity lemma*

Recall that the Bestvina-Brady group associated to a graph $\Gamma = (\mathsf{V}, \mathsf{E})$ is the kernel of the homomorphism $\nu\colon G_\Gamma \to \mathbb{Z}$ that sends each generator $v \in \mathsf{V}$ of G_Γ to $1 \in \mathbb{Z}$. Let $\iota\colon N_\Gamma \to G_\Gamma$ be the inclusion map.

Lemma 3.2. *If the graph Γ is connected, then the induced homomorphism $\iota_*\colon H_1(N_\Gamma) \to H_1(G_\Gamma)$ is injective.*

Proof. Let $\widetilde{C}_\bullet = (C_\bullet(\widetilde{K}_\Gamma), \partial_\bullet)$ be the Salvetti complex, with boundary maps given by (3.1). Identify the group ring $\mathbb{Z}\mathbb{Z}$ with the ring of Laurent polynomials $\mathbb{Z}[\tau, \tau^{-1}]$. By Shapiro's Lemma, $H_*(N_\Gamma) = H_*(\mathbb{Z}\mathbb{Z} \otimes_{\mathbb{Z}G_\Gamma} \widetilde{C}_\bullet)$, where $\mathbb{Z}\mathbb{Z} = \mathbb{Z}[\tau, \tau^{-1}]$ is viewed as a right G_Γ-module via the map $v \mapsto \tau$.

After identifying $\mathbb{Z}\mathbb{Z} \otimes_{\mathbb{Z}G_\Gamma} \widetilde{C}_k$ with $\mathbb{Z}\mathbb{Z} \otimes C_k$, the boundary map ∂_k takes the form $(\tau - 1) \otimes d_k\colon \mathbb{Z}\mathbb{Z} \otimes C_k \to \mathbb{Z}\mathbb{Z} \otimes C_{k-1}$, where d_k is the simplicial boundary map on the $(k-1)$-simplices of Δ_Γ. With these identifications, the chain map $\iota_\#\colon \mathbb{Z}\mathbb{Z} \otimes_{\mathbb{Z}G_\Gamma} C_\bullet(\widetilde{K}_\Gamma) \to C_\bullet(K_\Gamma)$ takes the form:

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{Z}\mathbb{Z} \otimes \mathbb{Z}^{\mathsf{E}} \arrow[r, "\partial_2 = (\tau-1)\otimes d_2"] \arrow[d, "\iota_2 = \epsilon \otimes \mathrm{id}"] & \mathbb{Z}\mathbb{Z} \otimes \mathbb{Z}^{\mathsf{V}} \arrow[r, "\partial_1 = (\tau-1)\otimes d_1"] \arrow[d, "\iota_1 = \epsilon \otimes \mathrm{id}"] & \mathbb{Z}\mathbb{Z} \otimes \mathbb{Z} \arrow[d, "\iota_0 = \epsilon \otimes \mathrm{id}"] \\ \cdots \arrow[r] & \mathbb{Z} \otimes \mathbb{Z}^{\mathsf{E}} \arrow[r, "0"] & \mathbb{Z} \otimes \mathbb{Z}^{\mathsf{V}} \arrow[r, "0"] & \mathbb{Z} \otimes \mathbb{Z} \end{tikzcd}$$

The map $\iota_*\colon H_1(N_\Gamma) \to H_1(G_\Gamma)$ is the homomorphism induced on H_1 by the middle down arrow. To show ι_* is injective, we need to prove: $\ker \partial_1 \cap \ker \iota_1 \subset \operatorname{im} \partial_2$.

Let $z = \sum_{v\in\mathsf{V}} p_v \otimes v \in \mathbb{Z}\mathbb{Z} \otimes \mathbb{Z}^{\mathsf{V}}$. Suppose z belongs to $\ker \partial_1$. Since $\partial_1(z) = (\tau - 1)(\sum_{v\in\mathsf{V}} p_v)$, and since the ring $\mathbb{Z}\mathbb{Z}$ has no zero-divisors, we must have $\sum_{v\in\mathsf{V}} p_v = 0$. Now suppose z belongs to $\ker \iota_1$. Then $\sum_{v\in\mathsf{V}} \epsilon(p_v) \otimes v = 0$, which can only happen if $\epsilon(p_v) = 0$, for all $v \in \mathsf{V}$. Thus, for each $v \in \mathsf{V}$, there is $q_v \in \mathbb{Z}\mathbb{Z}$ such that

$p_v = (\tau - 1)q_v$. We conclude that $\ker \partial_1 \cap \ker \iota_1$ is generated by elements of the form $(\tau - 1)q \otimes (v - u)$.

Let $e_1, \dots, e_s$ be a path in Γ with $e_i = (u_{i-1}, u_i)$, joining $u_0 = u$ to $u_s = v$. Then:

$$\partial_2\Big(\sum_{i=1}^{s} q \otimes e_i\Big) = (\tau - 1)\sum_{i=1}^{s} q \otimes (u_i - u_{i-1}) = (\tau - 1)q \otimes (v - u).$$

This finishes the proof. □

4. *Alexander invariants*

Let G be a group, with abelianization $H_1(G) = G/G'$. The *Alexander invariant* of G is the quotient group $B(G) = G'/G''$, endowed with the $\mathbb{Z}H_1(G)$-module structure induced by conjugation in G/G'', via the exact sequence $0 \to G'/G'' \to G/G'' \to G/G' \to 0$. Alternatively, if K is a connected CW-complex with $\pi_1(K) = G$, and K^{ab} is the universal abelian cover of K, then $B(G) = H_1(K^{\mathrm{ab}})$, with module structure coming from the action of $H_1(G)$ by deck transformations.

In this Section, we determine the Alexander invariants of right-angled Artin groups and of finitely-generated Bestvina-Brady groups. We start by giving a finite presentation for $B(G_\Gamma)$, viewed as a module over $\mathbb{Z}H_1(G_\Gamma)$, for an arbitrary finite graph $\Gamma = (\mathsf{V}, \mathsf{E})$.

After fixing a total ordering on V, we may identify $\mathbb{Z}H_1(G_\Gamma)$ with the ring of Laurent polynomials in variables labeled by the vertices, $\Lambda = \mathbb{Z}[\mathsf{V}^{\pm 1}]$. Let t be a triple of vertices, and let e be a 2-element subset of t. We denote by v_e the third vertex of t, and by ϵ_e the sign of the permutation (v_e, u, w) where $e = \{u, w\}$, with $u < w$.

Theorem 4.1. *The Alexander invariant $B(G_\Gamma)$ is the $\mathbb{Z}[\mathsf{V}^{\pm 1}]$-module generated by the non-edges $e \in \mathsf{E}_{\overline{\Gamma}}$, and with relators*

$$\sum_{e \subset t,\, e \in \mathsf{E}_{\overline{\Gamma}}} \epsilon_e (v_e - 1) \otimes e,$$

indexed by the triples of vertices $t \in \binom{\mathsf{V}}{3}$ which are not 3-cliques of Γ.

Proof. As before, let $\widetilde{C}_\bullet = (C_\bullet(\widetilde{K}_\Gamma), \partial_\bullet)$ be the equivariant chain complex of K_Γ. By Shapiro's Lemma, $B(G_\Gamma) = H_1(\mathbb{Z}H_1(G_\Gamma) \otimes_{\mathbb{Z}G_\Gamma} \widetilde{C}_\bullet)$, as Λ-modules.

Recall K_Γ is a CW-subcomplex of the torus T^n, where $n = |\mathsf{V}|$. Since $T^n = K(\mathbb{Z}^n, 1)$, the equivariant chain complex $(C_\bullet(\widetilde{T}^n), \delta_\bullet)$ gives a free Λ-resolution of $\mathbb{Z}$. Using (3.1), it is readily seen that $\Lambda \otimes_{\mathbb{Z}G_\Gamma} \widetilde{C}_\bullet = \Lambda \otimes C_\bullet$ is a Λ-subcomplex of $C_\bullet(\widetilde{T}^n) = \Lambda^{\binom{\mathsf{V}}{\bullet}}$, and that these two complexes coincide up to degree $\bullet = 1$. A diagram chase yields

$$B(G_\Gamma) = \operatorname{coker}\left(\delta_3 + \text{incl: } \Lambda^{\binom{\mathsf{V}}{3}} \oplus \Lambda^{\mathsf{E}} \to \Lambda^{\binom{\mathsf{V}}{2}}\right). \tag{4.1}$$

Row-reducing the above presentation matrix finishes the proof. □

A homomorphism $\iota\colon N \to G$ induces in a natural way a change of rings map $\iota_*\colon \mathbb{Z}H_1(N) \to \mathbb{Z}H_1(G)$ and a $\mathbb{Z}H_1(N)$-linear map $B(\iota)\colon B(N) \to B(G)$.

PROPOSITION 4.2. *Let Γ be a connected graph, and let $\iota\colon N_\Gamma \to G_\Gamma$ be the inclusion map. Then, the following hold.*

(i) *There is a split exact sequence* $0 \longrightarrow H_1(N_\Gamma) \xrightarrow{\iota_*} H_1(G_\Gamma) \xrightarrow{\nu_*} \mathbb{Z} \longrightarrow 0$.
(ii) *The inclusion ι restricts to an equality $N'_\Gamma = G'_\Gamma$.*
(iii) *The induced map $B(\iota)\colon B(N_\Gamma) \to B(G_\Gamma)$ is a $\mathbb{Z}H_1(N_\Gamma)$-linear isomorphism.*

Proof. ((i)) This a direct consequence of Lemma 3.2.
((ii)) Clearly, $N'_\Gamma \subset G'_\Gamma$. Now let $g \in G'_\Gamma$; then $\nu(g) = 0$, and so $g = \iota(n)$, for some $n \in N_\Gamma$. Since $\iota_*\colon H_1(N_\Gamma) \to H_1(G_\Gamma)$ is injective, we must have $n \in N'_\Gamma$.
((iii)) By the above, $N''_\Gamma = G''_\Gamma$, and so $N'_\Gamma/N''_\Gamma = G'_\Gamma/G''_\Gamma$. ☐

COROLLARY 4.3. *Let Γ be a connected graph, with vertex set $\mathsf{V} = \{1, \dots, n\}$. The Alexander invariant $B(N_\Gamma)$ is isomorphic to the restriction of the $\mathbb{Z}\mathbb{Z}^n$-module $B(G_\Gamma)$, with presentation given in Theorem 4.1, via the change of rings $\iota_*\colon \mathbb{Z}\mathbb{Z}^{n-1} \to \mathbb{Z}\mathbb{Z}^n$.*

Let G be a finitely presented group, with torsion-free abelianization. The *holonomy Lie algebra* of G, denoted $\mathfrak{H}(G)$, is the quotient of the free Lie algebra on $H_1(G)$ by the ideal generated by the image of the comultiplication map $\nabla_G\colon H_2(G) \to H_1(G) \wedge H_1(G)$.

The *infinitesimal Alexander invariant* of G is $\mathfrak{B}(G) = \mathfrak{H}(G)'/\mathfrak{H}(G)''$, with module structure over the symmetric algebra $\mathrm{Sym}(H_1(G))$ coming from the exact sequence

$$0 \to \mathfrak{H}(G)'/\mathfrak{H}(G)'' \to \mathfrak{H}(G)/\mathfrak{H}(G)'' \to \mathfrak{H}(G)/\mathfrak{H}(G)' \to 0.$$

This module is isomorphic to the "linearization" of the classical Alexander invariant of the group G, see [**21**].

A homomorphism $\iota\colon N \to G$ induces a change of rings map $\iota_*\colon \mathrm{Sym}(H_1(N)) \to \mathrm{Sym}(H_1(G))$ and a $\mathrm{Sym}(H_1(N))$-linear map $\mathfrak{B}(\iota)\colon \mathfrak{B}(N) \to \mathfrak{B}(G)$.

Now let $\Gamma = (\mathsf{V}, \mathsf{E})$ be a finite graph. After fixing a total ordering on V, we may identify $\mathrm{Sym}(H_1(G_\Gamma))$ with the polynomial ring $S = \mathbb{Z}[\mathsf{V}]$. Using [**21**, Theorem 6.2], we obtain the following infinitesimal analogue of Theorem 4.1.

PROPOSITION 4.4. *The infinitesimal Alexander invariant of a right-angled Artin group, $\mathfrak{B}(G_\Gamma) = \mathfrak{H}(G_\Gamma)'/\mathfrak{H}(G_\Gamma)''$, is the S-module generated by the non-edges $e \in \mathsf{E}_{\overline{\Gamma}}$, and with relators*

$$\sum_{e \subset t,\, e \in \mathsf{E}_{\overline{\Gamma}}} \epsilon_e v_e \otimes e,$$

indexed by the triples $t \in \binom{\mathsf{V}}{3}$ which are not 3-cliques of Γ.

5. *Lower central series*

In this section, we determine the associated graded Lie algebra and the Chen Lie algebra of the Bestvina-Brady group corresponding to a finite, connected graph, thus proving Theorems 1.1 and 1.2 from the Introduction.

5.1. *Lie algebras associated to right-angled Artin groups*

A graph $\Gamma = (\mathsf{V}, \mathsf{E})$ determines in a natural way a graded, finitely-presented Lie algebra $\mathfrak{H}_\Gamma$, as follows:

$$\mathfrak{H}_\Gamma = \mathrm{Lie}(\mathsf{V})/([v,w]=0 \text{ if } \{v,w\} \in \mathsf{E}), \tag{5.1}$$

where $\mathrm{Lie}(\mathsf{V})$ is the free Lie algebra on the vertex set V.

For each $k \geq 1$, let $f_k(\Gamma)$ be the number of complete k-subgraphs of Γ, and set $f_0(\Gamma) = 1$.

Theorem 5.1 ([**8**], [**9**], [**22**]). *Let Γ be a finite graph, and let G_Γ be the corresponding right-angled Artin group. Then $\mathrm{gr}(G_\Gamma) \cong \mathfrak{H}_\Gamma$, as graded Lie algebras. Moreover, the graded pieces of $\mathrm{gr}(G_\Gamma)$ are torsion-free, with ranks $\phi_k = \mathrm{rank}(\mathrm{gr}_k(G_\Gamma))$ given by:*

$$\prod_{k=1}^{\infty} (1-t^k)^{\phi_k} = P_\Gamma(-t),$$

where $P_\Gamma(t) = \sum_{k\geq 0} f_k(\Gamma) t^k$ is the clique polynomial of Γ.

For each $j \geq 1$, let $c_j(\Gamma) = \sum_{\mathsf{W}\subset \mathsf{V}\colon |\mathsf{W}|=j} \tilde{b}_0(\Gamma_{\mathsf{W}})$, where $\tilde{b}_0(\Gamma) = \mathrm{rank}\, \widetilde{H}_0(\Gamma)$ is the number of components of Γ minus 1. Note that $c_1(\Gamma) = 0$, and also $c_j(\Gamma) = 0$, if $j > |\mathsf{V}| - \kappa(\Gamma)$, where $\kappa(\Gamma)$ is the connectivity of Γ.

Theorem 5.2 [**22**]. *Let Γ be a finite graph. Then $\mathrm{gr}(G_\Gamma/G''_\Gamma) \cong \mathfrak{H}_\Gamma/\mathfrak{H}''_\Gamma$, as graded Lie algebras. Moreover, the graded pieces of $\mathrm{gr}(G_\Gamma/G''_\Gamma)$ are torsion-free, with ranks $\theta_k = \mathrm{rank}(\mathrm{gr}_k(G_\Gamma/G''_\Gamma))$ given by:*

$$\sum_{k=2}^{\infty} \theta_k t^k = Q_\Gamma\Big(\frac{t}{1-t}\Big),$$

where $Q_\Gamma(t) = \sum_{j=2}^{|\mathsf{V}|-\kappa(\Gamma)} c_j(\Gamma) t^j$ is the cut polynomial of Γ.

5.2. *Monodromy action*

Let N_Γ be the Bestvina-Brady group associated to the graph Γ. Recall we have an exact sequence

$$1 \longrightarrow N_\Gamma \xrightarrow{\ \iota\ } G_\Gamma \xrightarrow{\ \nu\ } \mathbb{Z} \longrightarrow 0\ . \tag{5.2}$$

This sequence admits a splitting $s\colon \mathbb{Z} \to G_\Gamma$, given by $s(1) = v$, for some fixed generator $v \in \mathsf{V}$.

Proposition 5.3. *Let Γ be a connected graph. Then, in the split extension (5.2), the group $\mathbb{Z}$ acts trivially on the abelianization $H_1(N_\Gamma)$.*

Proof. The monodromy of the semidirect product $G_\Gamma = N_\Gamma \rtimes_\sigma \mathbb{Z}$ is given by $\sigma(1)(x) = vxv^{-1}$, for some fixed generator $v \in \mathsf{V}$. Conjugation by any element of G_Γ acts trivially on $H_1(G_\Gamma)$. On the other hand, we know from Lemma 3.2 that $H_1(N_\Gamma)$ injects into $H_1(G_\Gamma)$. Hence, conjugation by an element of G_Γ also acts trivially on $H_1(N_\Gamma)$. ∎

For a homomorphism $\alpha\colon G \to H$, let $\bar{\alpha}\colon G/G'' \to H/H''$ be the induced homomorphism on maximal metabelian quotients.

PROPOSITION 5.4. *Let Γ be a connected graph. Then, the sequence*

$$1 \longrightarrow N_\Gamma/N''_\Gamma \xrightarrow{\bar{\iota}} G_\Gamma/G''_\Gamma \xrightarrow{\bar{\nu}} \mathbb{Z} \longrightarrow 0\ , \tag{5.3}$$

is split exact, with trivial monodromy action on $H_1(N_\Gamma/N''_\Gamma) = H_1(N_\Gamma)$.

Proof. From the proof of Proposition 4.2, we know that $N''_\Gamma = G''_\Gamma$. All claimed properties of sequence (5.3) follow from the corresponding properties of (5.2). □

5.3. *Lie algebras associated to Bestvina-Brady groups*

Recall now the following well-known result of Falk and Randell.

THEOREM 5.5 ([**11**]). *Let $1 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 1$ be a split exact sequence of groups. Suppose C acts trivially on $H_1(A)$. Then*

$$0 \longrightarrow \mathrm{gr}(A) \xrightarrow{\mathrm{gr}(\alpha)} \mathrm{gr}(B) \xrightarrow{\mathrm{gr}(\beta)} \mathrm{gr}(C) \longrightarrow 0$$

is a split exact sequence of graded Lie algebras.

This result permits us to reduce the computation of the LCS and Chen quotients of a Bestvina-Brady group to the computation of the LCS and Chen quotients of the corresponding right-angled Artin group.

THEOREM 5.6. *Let Γ be a finite, connected graph. Then, the inclusion map $\iota\colon N_\Gamma \to G_\Gamma$ induces isomorphisms of graded Lie algebras*
(i) $\mathrm{gr}'(\iota)\colon \mathrm{gr}'(N_\Gamma) \xrightarrow{\cong} \mathrm{gr}'(G_\Gamma)$.
(ii) $\mathrm{gr}'(\bar{\iota})\colon \mathrm{gr}'(N_\Gamma/N''_\Gamma) \xrightarrow{\cong} \mathrm{gr}'(G_\Gamma/G''_\Gamma)$.

Proof. Using Propositions 5.3 and 5.4, we may apply Theorem 5.5 to the exact sequences (5.2) and (5.3). Noting that $\mathrm{gr}(\mathbb{Z}) = \mathbb{Z}$ (concentrated in degree 1), yields isomorphisms (i) and (ii), respectively. □

Combining Theorem 5.6 with Theorems 5.1 and 5.2 finishes the proof of Theorems 1.1 and 1.2 from the Introduction.

EXAMPLE 5.7. Suppose Γ is a tree on n vertices. Then recall N_Γ is a free group of rank $n-1$. By Theorem 5.6, $\phi_k(G_\Gamma) = \phi_k(F_{n-1})$ and $\theta_k(G_\Gamma) = \theta_k(F_{n-1})$, for all $k \geq 2$, which recovers the computations from [**22**, §6.2].

REMARK 5.8. Suppose $\Gamma = K_{n_1,\dots,n_r}$ is a complete multi-partite graph. Then $G_\Gamma = F_{n_1} \times \dots \times F_{n_r}$, and so, by Theorem 5.6, $\mathrm{gr}'(N_\Gamma) = \mathrm{gr}'(F_{n_1} \times \dots \times F_{n_r})$. Even though, from the point of view of the lower central series quotients, N_Γ looks like a product of free groups, it is not of this type, except when some $n_i = 1$. Indeed, if all $n_i > 1$, then $H_{r-1}(\Delta_\Gamma)$ is a free abelian group of rank $\prod_{i=1}^r (n_i - 1) > 0$, and so it follows from [**1**] that N_Γ does not have a finite $K(N_\Gamma, 1)$.

6. *Holonomy Lie algebra and* 1*-formality*

Let G be a finitely presented group, with torsion-free abelianization. Recall that the holonomy Lie algebra $\mathfrak{H}(G)$ is the quotient of the free Lie algebra $\mathrm{Lie}(H_1(G))$ by the ideal generated by the image of the comultiplication map $\nabla_G\colon H_2(G) \to H_1(G) \wedge H_1(G)$. Note that $\mathfrak{H}(G)$ inherits a natural grading from the free Lie algebra, compatible with the Lie bracket. By construction, $\mathfrak{H}(G)$ is generated by $\mathfrak{H}_1(G)$. Consequently, the derived Lie subalgebra, $\mathfrak{H}'(G)$, coincides with $\mathfrak{H}_{\geq 2}(G)$. If we drop the torsion-freeness assumption on $H_1(G)$, we may still define the rational holonomy Lie algebra, $\mathfrak{H}_{\mathbb{Q}}(G)$, using the rational homology groups of G.

To a group G, Quillen associates in a functorial way a Malcev filtered Lie algebra, M_G; see [**24**, Appendix A], and also [**21**] for further details. A finitely presented group G is said to be 1-*formal* if M_G is isomorphic to the rational holonomy Lie algebra, $\mathfrak{H}_{\mathbb{Q}}(G)$, completed with respect to the bracket length filtration. Equivalently, M_G is a quadratic Malcev Lie algebra. If the group G is 1-formal, then $\mathrm{gr}(G) \otimes \mathbb{Q} \cong \mathfrak{H}_{\mathbb{Q}}(G)$, as follows from [**24**]. Moreover, $\mathrm{gr}(G/G'') \otimes \mathbb{Q} \cong \mathfrak{H}_{\mathbb{Q}}(G)/\mathfrak{H}_{\mathbb{Q}}(G)''$, as shown in [**21**].

Assume now G is finitely presented, and $H_1(G)$ is torsion-free. Then, the canonical projection $\mathrm{Lie}(H_1(G)) \twoheadrightarrow \mathrm{gr}(G)$ factors through an epimorphism of graded Lie algebras, $\Psi_G\colon \mathfrak{H}(G) \twoheadrightarrow \mathrm{gr}(G)$, which in turn descends to an epimorphism

$$\Psi_G^{(2)}\colon \mathfrak{H}(G)/\mathfrak{H}(G)'' \twoheadrightarrow \mathrm{gr}(G/G'').$$

If the group G is 1-formal, then the maps $\Psi_G \otimes \mathbb{Q}$ and $\Psi_G^{(2)} \otimes \mathbb{Q}$ are isomorphisms, see [**21**].

By construction, the resonance variety $\mathcal{R}_1(G)$ depends only on the holonomy Lie algebra $\mathfrak{H}_{\mathbb{Q}}(G)$. More precisely, if $\mathfrak{H}_{\mathbb{Q}}(G_1) \cong \mathfrak{H}_{\mathbb{Q}}(G_2)$, as graded Lie algebras, then there is a linear isomorphism $H^1(G_1, \mathbb{C}) \cong H^1(G_2, \mathbb{C})$, restricting to an isomorphism $\mathcal{R}_1(G_1) \cong \mathcal{R}_1(G_2)$.

For a right-angled Artin group G_Γ, it is easily seen that $\mathfrak{H}(G_\Gamma) = \mathfrak{H}_\Gamma$, cf. [**22**]. Moreover, as shown by Kapovich and Millson [**13**], the group G_Γ is 1-formal. We now prove an analogous result for the Bestvina-Brady groups.

Proposition 6.1. *If the flag complex Δ_Γ is simply-connected, then N_Γ is 1-formal.*

Proof. Consider the Dicks-Leary presentation (2.3) for N_Γ. It follows from [**19**] that the Malcev Lie algebra of N_Γ is the quotient of $\widehat{\mathrm{Lie}(\mathsf{E})}$, the free Malcev Lie algebra on E, by the closed Lie ideal generated by the elements $(e,f), (f,g), (e,g), efg^{-1}$, for all directed triangles (e,f,g) as in Figure 1. Here multiplication denotes the Campbell-Hausdorff product in the underlying Malcev group.

Now use [**20**, Lemma 2.5] to replace the CH commutators $(e,f), (f,g), (e,g)$ by the corresponding Lie commutators, $[e,f], [f,g], [e,g]$. It follows from the definition of the CH product that we may also replace efg^{-1} by $e+f-g$. This shows that M_{N_Γ} is a quadratic Malcev Lie algebra, and so, N_Γ is 1-formal. $\square$

Let $\iota\colon N \to G$ be a homomorphism between finitely presented groups with torsion-free abelianizations. Denote by $\iota_*\colon H_*(N) \to H_*(G)$ the induced homomorphism in homology. We then have a commuting diagram,

$$\begin{tikzcd} H_2(N) \arrow[r, "\nabla_N"] \arrow[d, "\iota_*"] & H_1(N) \wedge H_1(N) \arrow[d, "\iota_* \wedge \iota_*"] \\ H_2(G) \arrow[r, "\nabla_G"] & H_1(G) \wedge H_1(G) \end{tikzcd} \tag{6.1}$$

Consequently, there is an induced morphism of graded Lie algebras, $\mathfrak{H}(\iota)\colon \mathfrak{H}(N) \to \mathfrak{H}(G)$.

Lemma 6.2. *If $\pi_1(\Delta_\Gamma) = 0$, then the map $\mathfrak{H}'_{\mathbb{Q}}(\iota)\colon \mathfrak{H}'_{\mathbb{Q}}(N_\Gamma) \to \mathfrak{H}'_{\mathbb{Q}}(G_\Gamma)$ is an isomorphism of graded Lie algebras.*

Proof. The inclusion $\iota\colon N_\Gamma \to G_\Gamma$ induces Lie algebra maps $\mathfrak{H}(\iota)\colon \mathfrak{H}(N_\Gamma) \to \mathfrak{H}(G_\Gamma)$ and $\mathrm{gr}(\iota)\colon \mathrm{gr}(N_\Gamma) \to \mathrm{gr}(G_\Gamma)$, which commute with the natural surjections from the holonomy to the associated graded Lie algebras. Passing to derived Lie subalgebras, and tensoring with $\mathbb{Q}$, we obtain the following commuting diagram:

$$\begin{tikzcd} \mathfrak{H}'(N_\Gamma) \otimes \mathbb{Q} \arrow[r, "\mathfrak{H}'(\iota)\otimes\mathbb{Q}"] \arrow[d, "\Psi_{N_\Gamma}\otimes\mathbb{Q}"] & \mathfrak{H}'(G_\Gamma) \otimes \mathbb{Q} \arrow[d, "\Psi_{G_\Gamma}\otimes\mathbb{Q}"] \\ \mathrm{gr}'(N_\Gamma) \otimes \mathbb{Q} \arrow[r, "\mathrm{gr}'(\iota)\otimes\mathbb{Q}"] & \mathrm{gr}'(G_\Gamma) \otimes \mathbb{Q} \end{tikzcd} \tag{6.2}$$

The vertical arrows are isomorphisms, by the 1-formality of G_Γ and N_Γ, insured by [**13**] and Proposition 6.1, respectively. The bottom arrow is an isomorphism, by Theorem 5.6(i). Hence, the top arrow, $\mathfrak{H}'(\iota) \otimes \mathbb{Q} = \mathfrak{H}'_{\mathbb{Q}}(\iota)$, is also an isomorphism. □

7. *Cohomology ring*

In this section, we exploit the 1-formality property of a finitely presented Bestvina-Brady group N_Γ, in order to give a purely combinatorial description of $H^{\le 2}(N_\Gamma, \mathbb{Q})$.

7.1. *Homology of N_Γ*

Fix a coefficient field $\Bbbk$. The extension $1 \to N_\Gamma \xrightarrow{\iota} G_\Gamma \xrightarrow{\nu} \mathbb{Z} \to 0$ defines a natural $\Bbbk\mathbb{Z}$-module structure on $H_*(N_\Gamma, \Bbbk)$. The next result gives a combinatorial description of this structure. (See also [**12**], [**16**] for related computations.)

Proposition 7.1. *Let Γ be a finite graph, with flag complex Δ_Γ. For each $r > 0$, we have an isomorphism of $\Bbbk\mathbb{Z}$-modules,*

$$H_r(N_\Gamma, \Bbbk) \cong (\Bbbk\mathbb{Z})^{\dim \widetilde{H}_{r-1}(\Delta_\Gamma, \Bbbk)} \oplus ({}_\epsilon\Bbbk)^{\dim B_{r-1}(\Delta_\Gamma, \Bbbk)},$$

where ${}_\epsilon\Bbbk$ denotes the trivial $\Bbbk\mathbb{Z}$-module $\Bbbk$, and $B_\bullet(\Delta_\Gamma, \Bbbk)$ are the simplicial boundaries.

Proof. By Shapiro's Lemma, $H_*(N_\Gamma, \Bbbk) = H_*(\Bbbk\mathbb{Z} \otimes_{\mathbb{Z}G_\Gamma} \widetilde{C}_\bullet)$, where $\widetilde{C}_\bullet$ is the equivariant chain complex from Proposition 3.1, and the change of rings $\mathbb{Z}G_\Gamma \to \Bbbk\mathbb{Z}$

is induced by $\nu\colon G_\Gamma \to \mathbb{Z}$. Write $P := \Bbbk\mathbb{Z} = \Bbbk[\tau^{\pm 1}]$. Using (3.1), we find that

$$H_r(N_\Gamma, \Bbbk) = \frac{P \otimes Z_{r-1}(\Delta_\Gamma, \Bbbk)}{(\tau - 1)P \otimes B_{r-1}(\Delta_\Gamma, \Bbbk)}, \tag{7.1}$$

as modules over P, where $Z_\bullet(\Delta_\Gamma, \Bbbk)$ denotes the reduced simplicial cycles.

Set $B := B_{r-1}(\Delta_\Gamma, \Bbbk)$, $Z := Z_{r-1}(\Delta_\Gamma, \Bbbk)$, and $H := \widetilde{H}_{r-1}(\Delta_\Gamma, \Bbbk)$. It is straightforward to check that the natural maps $\Bbbk \otimes B \hookrightarrow P \otimes Z$ and $P \otimes Z \twoheadrightarrow P \otimes H$ give rise to the following split exact sequence of P-modules:

$$0 \longrightarrow {}_\epsilon\Bbbk \otimes B \longrightarrow \frac{P \otimes Z}{(\tau - 1)P \otimes B} \longrightarrow P \otimes H \longrightarrow 0\ . \tag{7.2}$$

The conclusion follows by putting together (7.1) and (7.2). $\square$

7.2. *Cohomology ring in low degrees*

Recall from Section 3 that $K_\Gamma = K(G_\Gamma, 1)$ is a subcomplex of the standard torus $(S^1)^{\mathsf{V}}$. This readily implies that the cohomology ring $H^*(G_\Gamma, \Bbbk)$ is the quotient of the exterior $\Bbbk$-algebra on V by the ideal generated by the monomials vw corresponding to non-edges $\{v, w\} \in \overline{\mathsf{E}}$.

Lemma 7.2. *If $\pi_1(\Delta_\Gamma) = 0$, then the following hold.*

(i) *The cup-product map $\cup_{N_\Gamma}\colon \bigwedge^2 H^1(N_\Gamma, \mathbb{Q}) \to H^2(N_\Gamma, \mathbb{Q})$ is surjective.*
(ii) *The map $\iota^*\colon H^2(G_\Gamma, \mathbb{Q}) \to H^2(N_\Gamma, \mathbb{Q})$ is surjective.*

Proof. (i) It is enough to show that $\dim_{\mathbb{Q}} H_2(N_\Gamma, \mathbb{Q}) = \dim_{\mathbb{Q}} \operatorname{im}(\nabla_{N_\Gamma})$. We know from Proposition 7.1 that $\dim H_2(N_\Gamma, \mathbb{Q}) = \dim Z_1(\Delta_\Gamma, \mathbb{Q}) = |\mathsf{E}| - |\mathsf{V}| + 1$, since Δ_Γ is simply-connected. On the other hand,

$$\begin{aligned}\dim \operatorname{im}(\nabla_{N_\Gamma}) &= \dim {\textstyle\bigwedge^2} H_1(N_\Gamma, \mathbb{Q}) - \dim \mathfrak{H}_2(N_\Gamma) \otimes \mathbb{Q} \\ &= \tbinom{|\mathsf{V}|-1}{2} - \Big(\tbinom{|\mathsf{V}|}{2} - |\mathsf{E}|\Big) \\ &= |\mathsf{E}| - |\mathsf{V}| + 1,\end{aligned}$$

by the definition of the holonomy Lie algebra, Lemma 6.2, and the injectivity of ∇_{G_Γ}.

(ii) Follows from Part (i), since we know from Lemma 3.2 that $\iota^*\colon H^1(G_\Gamma, \mathbb{Q}) \to H^1(N_\Gamma, \mathbb{Q})$ is surjective. $\square$

7.3. *Proof of Theorem 1.3*

Clearly, ι^* factors through the quotient by the ideal generated by ν, since $\nu\iota = 0$. The isomorphism claim in degree 1 follows immediately from Proposition 4.2(i). The surjectivity property in degree 2 is a direct consequence of Lemma 7.2(ii). We are left with proving that

$$\ker\big(\iota^*\colon H^2(G_\Gamma, \mathbb{Q}) \to H^2(N_\Gamma, \mathbb{Q})\big) \subset \operatorname{im}\big(\cdot\nu\colon H^1(G_\Gamma, \mathbb{Q}) \to H^2(G_\Gamma, \mathbb{Q})\big)\,.$$

By dualizing, it is enough to check that the inclusion

$$\ker\big(\mu_\nu^\top \circ \nabla_{G_\Gamma}\colon H_2(G_\Gamma, \mathbb{Q}) \to H_1(G_\Gamma, \mathbb{Q})\big) \subset \operatorname{im}\big(\iota_*\colon H_2(N_\Gamma, \mathbb{Q}) \to H_2(G_\Gamma, \mathbb{Q})\big),$$

holds, where $\mu_\nu^\top$ is the transpose of $\mu_\nu\colon H^1(G_\Gamma, \mathbb{Q}) \to \bigwedge^2 H^1(G_\Gamma, \mathbb{Q})$, the right-multiplication by ν. It follows from Proposition 4.2(i) that $\ker\big(\mu_\nu^\top\big) = \operatorname{im}\big(\bigwedge^2 \iota_*\big)$.

Hence, $\ker\big(\mu_\nu^\top \circ \nabla_{G_\Gamma}\big) = \nabla_{G_\Gamma}^{-1}\big(\operatorname{im}\big(\bigwedge^2 \iota_*\big)\big)$. Now recall from Lemma 6.2 that $\bigwedge^2 \iota_*$ induces an isomorphism

$$\bigwedge^2 \iota_*\colon \bigwedge^2 H_1(N_\Gamma,\mathbb{Q})/\operatorname{im}(\nabla_{N_\Gamma}) \xrightarrow{\cong} \bigwedge^2 H_1(G_\Gamma,\mathbb{Q})/\operatorname{im}(\nabla_{G_\Gamma}). \tag{7.3}$$

The desired inclusion follows at once from (7.3) and diagram (6.1).

This finishes the proof of Theorem 1.3 from the Introduction. In [**16**], Leary and Saadetoğlu, using a different approach, obtain a similar description of the truncated cohomology ring $H^{\le r}(N_\Gamma)$, in the situation when $\widetilde{H}_{<r}(\Delta_\Gamma)=0$.

8. *Characteristic and resonance varieties*

In previous work [**22**], [**5**], we determined the resonance and characteristic varieties of right-angled Artin groups. In this section, we do the same for finitely presented Bestvina-Brady groups, thus proving Theorem 1.4 from the Introduction.

8.1. *Jumping loci for G_Γ*

Let $\Gamma=(\mathsf{V},\mathsf{E})$ be a finite graph, and let G_Γ be the corresponding right-angled Artin group. Write $H_{\mathsf{V}}=H^1(G_\Gamma,\mathbb{C})$ and $\mathbb{T}_{\mathsf{V}}=\operatorname{Hom}(G_\Gamma,\mathbb{C}^*)$. If W is a subset of V, write H_{W} and $\mathbb{T}_{\mathsf{W}}$ for the corresponding coordinate subspaces, respectively, subtori.

Theorem 8.1 ([**22**], [**5**]). *Let Γ be a finite graph. Then:*

$$\mathcal{R}_1(G_\Gamma)=\bigcup_{\substack{\mathsf{W}\subset\mathsf{V}\\ \Gamma_{\mathsf{W}}\text{ disconnected}}} H_{\mathsf{W}} \quad\text{and}\quad \mathcal{V}_1(G_\Gamma)=\bigcup_{\substack{\mathsf{W}\subset\mathsf{V}\\ \Gamma_{\mathsf{W}}\text{ disconnected}}} \mathbb{T}_{\mathsf{W}}.$$

It follows that the irreducible components of the above varieties are indexed by the subsets $\mathsf{W}\subset\mathsf{V}$, maximal among those for which Γ_{W} is disconnected. In particular, if Γ is disconnected, then $\mathcal{R}_1(G_\Gamma)=H_{\mathsf{V}}$ and $\mathcal{V}_1(G_\Gamma)=\mathbb{T}_{\mathsf{V}}$.

Example 8.2. Let Γ be a tree on $n>2$ vertices. Then $\mathcal{R}_1(G_\Gamma)$ is a union of coordinate hyperplanes, one for each cut point (that is, non-extremal vertex) of Γ. In particular, the irreducible components of $\mathcal{R}_1(G_\Gamma)$ are in general position.

8.2. *A map between jumping loci*

Let $N_\Gamma=\ker(\nu\colon G_\Gamma\twoheadrightarrow\mathbb{Z})$, and let $\iota\colon N_\Gamma\hookrightarrow G_\Gamma$ be the natural inclusion.

Assuming Γ is connected, we infer from Proposition 4.2(i) that ι induces a vector space epimorphism $\iota^*\colon H^1(G_\Gamma,\mathbb{C})\twoheadrightarrow H^1(N_\Gamma,\mathbb{C})$. Identifying $H^1(G_\Gamma,\mathbb{C})$ with $\mathbb{C}^{\mathsf{V}}$, the kernel of ι^* gets identified with the diagonal line.

Similarly, ι induces an algebraic group epimorphism $\iota^*\colon\mathbb{T}_{G_\Gamma}\twoheadrightarrow\mathbb{T}_{N_\Gamma}$. Identifying $\mathbb{T}_{G_\Gamma}$ with $(\mathbb{C}^*)^{\mathsf{V}}$, the kernel of ι^* gets identified with the diagonal 1-dimensional subtorus.

Lemma 8.3. *If the flag complex Δ_Γ is simply-connected, and $|\mathsf{V}|>1$, then*

(i) *The map $\iota^*\colon H^1(G_\Gamma,\mathbb{C})\twoheadrightarrow H^1(N_\Gamma,\mathbb{C})$ restricts to a surjection $\iota^*\colon\mathcal{R}_1(G_\Gamma)\twoheadrightarrow\mathcal{R}_1(N_\Gamma)$.*

(ii) *The map $\iota^*\colon\mathbb{T}_{G_\Gamma}\twoheadrightarrow\mathbb{T}_{N_\Gamma}$ restricts to a surjection $\iota^*\colon\mathcal{V}_1(G_\Gamma)\twoheadrightarrow\mathcal{V}_1(N_\Gamma)$.*

Proof. Part (i). There is an intimate connection between the resonance variety and the infinitesimal Alexander invariant of a finitely presented group G. More precisely, $\mathcal{R}_1(G)$ coincides, away from the origin, with the zero set of the annihilator of the $\mathrm{Sym}(H_1(G))\otimes\mathbb{C}$-module $\mathfrak{B}(G)\otimes\mathbb{C}$. This follows from [**5**, Lemma 4.2], together with the description of radicals of Fitting ideals in terms of annihilators, see [**10**, pp. 511–513].

It follows from Lemma 6.2 that $\mathfrak{B}(\iota)\otimes\mathbb{C}\colon \mathfrak{B}(N_\Gamma)\otimes\mathbb{C}\to\mathfrak{B}(G_\Gamma)\otimes\mathbb{C}$ is an isomorphism of modules over $\mathrm{Sym}(H_1(N_\Gamma))\otimes\mathbb{C}$. Hence,

$$\mathrm{ann}(\mathfrak{B}(N_\Gamma)\otimes\mathbb{C}) = (\iota_*\otimes\mathbb{C})^{-1}(\mathrm{ann}(\mathfrak{B}(G_\Gamma)\otimes\mathbb{C})). \tag{8.1}$$

Taking the complex varieties defined by the ideals on both sides of (8.1) finishes the proof.

Part (ii). Similarly, we know from [**5**, Lemma 4.5] that $\mathcal{V}_1(G)$ coincides, away from the origin, with the zero set of the annihilator of the $\mathbb{C}H_1(G)$-module $B(G)\otimes\mathbb{C}$. By Proposition 4.2(iii), the map $B(\iota)\otimes\mathbb{C}\colon B(N_\Gamma)\otimes\mathbb{C}\to B(G_\Gamma)\otimes\mathbb{C}$ is an isomorphism of modules over $\mathbb{C}H_1(N_\Gamma)$. As above, we conclude by taking the complex varieties defined by the annihilators of these two modules. $\square$

Lemma 8.4. *Suppose* W *is a proper subset of the vertex set* V *of* Γ. *Then the restriction of* $\iota^*\colon H^1(G_\Gamma,\mathbb{C})\to H^1(N_\Gamma,\mathbb{C})$ *to the coordinate subspace* H_{W} *is injective. Similarly, the restriction of* $\iota^*\colon \mathbb{T}_{G_\Gamma}\to\mathbb{T}_{N_\Gamma}$ *to the coordinate subtorus* $\mathbb{T}_{\mathsf{W}}$ *is injective.*

Proof. Let $\{e_v\}_{v\in\mathsf{V}}$ be the standard basis for the vector space $H^1(G_\Gamma,\mathbb{C})=\mathbb{C}^{\mathsf{V}}$. Suppose $\iota^*\big(\sum_{w\in\mathsf{W}} c_w e_w\big)=0$. Then $\sum_{w\in\mathsf{W}} c_w e_w = c\sum_{v\in\mathsf{V}} e_v$, for some scalar $c\in\mathbb{C}$. Picking $v\in\mathsf{V}\setminus\mathsf{W}$, and comparing the coefficient of e_v on both sides of this equation, we see that $c=0$. This finishes the proof of the first claim; the proof of the second claim follows along the same lines. $\square$

For a subset $\mathsf{W}\subset\mathsf{V}$, let H'_{W} denote the subspace $\iota^*(H_{\mathsf{W}})\subset H^1(N_\Gamma,\mathbb{C})$, and let $\mathbb{T}'_{\mathsf{W}}$ denote the subtorus $\iota^*(\mathbb{T}_{\mathsf{W}})\subset\mathbb{T}_{N_\Gamma}$.

Lemma 8.5. *Suppose* W_1 *and* W_2 *are two subsets of* V, *of size at most* $|\mathsf{V}|-2$. *If* $H'_{\mathsf{W}_1}\subset H'_{\mathsf{W}_2}$, *or* $\mathbb{T}'_{\mathsf{W}_1}\subset\mathbb{T}'_{\mathsf{W}_2}$, *then* $\mathsf{W}_1\subset\mathsf{W}_2$.

Proof. Assume there is a vertex $v_1\in\mathsf{W}_1\setminus\mathsf{W}_2$. Since $|\mathsf{W}_2|\le|\mathsf{V}|-2$, there must be another vertex $v_2\in\mathsf{V}\setminus\mathsf{W}_2$, distinct from v_1. Suppose $H'_{\mathsf{W}_1}\subset H'_{\mathsf{W}_2}$. Then $e_{v_1}=\sum_{v\in\mathsf{W}_2} c_v e_v + c\sum_{v\in\mathsf{V}} e_v$. Comparing coefficients of e_{v_2} on both sides, we find $c=0$; hence $e_{v_1}\in H_{\mathsf{W}_2}$, a contradiction. The case $\mathbb{T}'_{\mathsf{W}_1}\subset\mathbb{T}'_{\mathsf{W}_2}$ is treated similarly. $\square$

8.3. *Proof of Theorem 1.4*

It follows from Theorem 8.1 and Lemma 8.3 that

$$\mathcal{R}_1(N_\Gamma)=\bigcup_{\mathsf{W}} H'_{\mathsf{W}} \qquad\text{and}\qquad \mathcal{V}_1(N_\Gamma)=\bigcup_{\mathsf{W}} \mathbb{T}'_{\mathsf{W}}, \tag{8.2}$$

where, in both cases, the union is taken over all subsets $\mathsf{W}\subset\mathsf{V}$, maximal among those for which Γ_{W} is disconnected. Lemma 8.4 guarantees that $H'_{\mathsf{W}}\subset H^1(N_\Gamma,\mathbb{C})$

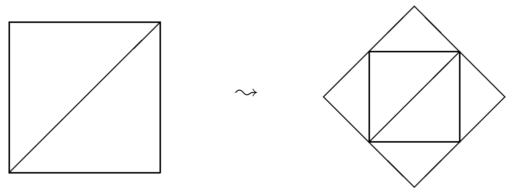

FIGURE 3. *Building an extra-special triangulation of the disk*

is a vector subspace of dimension $|\mathsf{W}|$, and $\mathbb{T}'_{\mathsf{W}} \subset \mathbb{T}_{N_\Gamma}$ is a subtorus of dimension $|\mathsf{W}|$.

First assume $\kappa(\Gamma) = 1$, that is, Γ has a cut point. This means there is a $v \in \mathsf{V}$ such that $\Gamma_{\mathsf{V}\setminus v}$ is disconnected. A dimension count shows that $\mathcal{R}_1(N_\Gamma) = H^1(N_\Gamma, \mathbb{C})$ and $\mathcal{V}_1(N_\Gamma) = \mathbb{T}_{N_\Gamma}$.

Now assume $\kappa(\Gamma) > 1$. We infer from Lemma 8.5 that (8.2) gives indeed the irreducible decompositions of the respective varieties. This ends the proof of Theorem 1.4.

EXAMPLE 8.6. Let Γ be a tree on $n > 2$ vertices. Then $\kappa(\Gamma) = 1$, and so $\mathcal{R}_1(N_\Gamma) = H^1(N_\Gamma, \mathbb{C}) = \mathbb{C}^{n-1}$, and $\mathcal{V}_1(N_\Gamma) = \mathbb{T}_{N_\Gamma} = (\mathbb{C}^*)^{n-1}$ (this computation also follows from the fact that $N_\Gamma = F_{n-1}$).

9. *Comparison with Artin groups and arrangement groups*

In this section, we use the methods developed above to compare the Bestvina-Brady groups to two other classes of groups: Artin groups and arrangement groups.

9.1. *Extra-special triangulations*

Recall we defined a triangulation Δ of the disk D^2 to be *special* if Δ can be obtained from a triangle by adding one triangle at a time, along a unique boundary edge.

LEMMA 9.1. *Let Δ be a special triangulation of the 2-disk, and $\Gamma = (\mathsf{V}, \mathsf{E})$ its 1-skeleton. Then $\mathcal{R}_1(N_\Gamma)$ is a proper subset of $H^1(N_\Gamma, \mathbb{C})$.*

Proof. Recall from Lemma 2.7(ii) that $\Delta_\Gamma = \Delta$; in particular, Theorem 1.4 applies. It is also readily seen that the graph Γ has no cut points, i.e, $\kappa(\Gamma) > 1$. Thus, $\mathcal{R}_1(N_\Gamma) \subsetneq H^1(N_\Gamma, \mathbb{C})$. ∎

DEFINITION 9.2. A triangulation of D^2 is called *extra-special* if it is obtained from a special triangulation, by adding one triangle along each boundary edge. (See Figure 3.)

If Δ is extra-special, more can be said about the resonance variety of N_Γ. By definition, Δ is obtained by attaching triangles to the boundary edges of a special triangulation of the disk. Denote by $(e_1, \dots, e_r)$ the circuit formed by these edges, and write $\mathsf{W}_i = \mathsf{V} \setminus e_i$. Note that each edge e_i is a minimal cut set of Γ; hence, H'_{W_i} is an irreducible component of $\mathcal{R}_1(N_\Gamma)$, for each $i = 1, \dots, r$.

LEMMA 9.3. *Let Γ be the 1-skeleton of an extra-special triangulation Δ of D^2. Then the subspace $\bigcap_{i=1}^r H'_{\mathsf{W}_i}$ has codimension $r-1$ in $H^1(N_\Gamma, \mathbb{C})$. In particular, $\bigcap_{i=1}^r H'_{\mathsf{W}_i} \neq 0$.*

Proof. We claim that

$$\iota^*\Big(\bigcap_{i=1}^r H_{\mathsf{W}_i}\Big) = \bigcap_{i=1}^r H'_{\mathsf{W}_i}. \tag{9.1}$$

The inclusion $\subseteq$ is clear. The reverse inclusion is proved by induction on s ($0 < s < r$), with the case $s = 1$ being obvious. Set $P_k = \bigcap_{i=1}^k H_{\mathsf{W}_i}$, $Q_k = H_{\mathsf{W}_k}$, $P'_k = \bigcap_{i=1}^k H'_{\mathsf{W}_i}$, and $Q'_k = H'_{\mathsf{W}_k}$. We then have the commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & P_s \cap Q_{s+1} \arrow[r] \arrow[d, "\iota^*"] & P_s \oplus Q_{s+1} \arrow[r] \arrow[d, "\iota^*"] & P_s + Q_{s+1} \arrow[r] \arrow[d, "\iota^*"] & 0 \\ 0 \arrow[r] & P'_s \cap Q'_{s+1} \arrow[r] & P'_s \oplus Q'_{s+1} \arrow[r] & P'_s + Q'_{s+1} \arrow[r] & 0 \end{tikzcd} \tag{9.2}$$

The middle arrow is an isomorphism, by Lemma 8.4 and the induction hypothesis. Clearly, the right arrow is an epimorphism. Note that $P_s + Q_{s+1}$ is a subspace of

$$H_{\mathsf{W}_1 \cap \dots \cap \mathsf{W}_s} + H_{\mathsf{W}_{s+1}} = H_{(\mathsf{W}_1 \cap \dots \cap \mathsf{W}_s) \cup \mathsf{W}_{s+1}}.$$

Since $(e_1 \cup \dots \cup e_s) \cap e_{s+1} \neq \emptyset$, this is a proper subspace of H_{V}. Thus, the right arrow in diagram (9.2) is injective, again by Lemma 8.4. Applying the 5-Lemma finishes the proof of the claim.

From (9.1), we see that

$$\operatorname{codim} \bigcap_{i=1}^r H'_{\mathsf{W}_i} = (|\mathsf{V}| - 1) - \dim \bigcap_{i=1}^r H_{\mathsf{W}_i} = r - 1.$$

Finally, if $\bigcap_{i=1}^r H'_{\mathsf{W}_i} = 0$, then $r = |\mathsf{V}|$. But clearly $|\mathsf{V}| \geq 2r$. ☐

9.2. *Artin groups*

A *weighted graph* is a graph $\Gamma = (\mathsf{V}, \mathsf{E})$ endowed with a function $m \colon \mathsf{E} \to \mathbb{Z}$ that assigns to each edge e an integer $m(e) \geq 2$. Such a weighted graph (Γ, m) determines an Artin group (of finite type),

$$G_{\Gamma,m} = \langle v \in \mathsf{V} \mid \pi_m(v,w) = \pi_m(w,v) \text{ if } \{v,w\} \in \mathsf{E} \rangle,$$

where $\pi_m(v,w) = vwv\dots$ has length $m(\{v,w\})$. When all edge weights are equal to 2, this is simply the right-angled Artin group G_Γ.

Associated to a weighted graph as above there is an ordinary (unlabeled) graph $\widetilde{\Gamma} = (\widetilde{\mathsf{V}}, \widetilde{\mathsf{E}})$, called the "odd contraction" of (Γ, m), see [**5**, §10.9]. First define Γ_{odd} to be the unlabeled graph with vertex set V and edge set $\{e \in \mathsf{E} \mid m(e) \text{ is odd}\}$. Then define $\widetilde{\mathsf{V}}$ to be the set of connected components of Γ_{odd}, with two distinct components determining an edge $\{c, c'\} \in \widetilde{\mathsf{E}}$ if and only if there exist vertices $v \in c$ and $v' \in c'$ which are connected by an edge in E.

Proposition 9.4. *Let Γ be the 1-skeleton of an extra-special triangulation of D^2. Then the Bestvina-Brady group N_Γ is not isomorphic to any Artin group.*

Proof. Suppose N_Γ is isomorphic to an Artin group $G_{\Gamma',m}$. Let $\widetilde{\Gamma}'$ be the odd contraction of Γ'. Lemma 10.11 from [**5**] guarantees that the respective Malcev Lie algebras, $M_{G_{\Gamma',m}}$ and $M_{G_{\widetilde{\Gamma}'}}$, are filtered isomorphic. On the other hand, we know from Theorem 16.10 from [**13**] that both Artin groups, $G_{\Gamma',m}$ and $G_{\widetilde{\Gamma}'}$, are 1-formal. Passing to associated graded Lie algebras, we obtain that $\mathfrak{H}_{\mathbb{Q}}(G_{\Gamma',m}) \cong \mathfrak{H}_{\mathbb{Q}}(G_{\widetilde{\Gamma}'})$, as graded Lie algebras. Hence, $\mathfrak{H}_{\mathbb{Q}}(N_\Gamma) \cong \mathfrak{H}_{\mathbb{Q}}(G_{\widetilde{\Gamma}'})$, as graded Lie algebras. This implies the existence of an ambient isomorphism

$$\mathcal{R}_1(N_\Gamma) \cong \mathcal{R}_1(G_{\widetilde{\Gamma}'}).$$

From Lemma 9.1, we know that $\mathcal{R}_1(N_\Gamma) \subsetneq H^1(N_\Gamma, \mathbb{C})$. By Theorem 8.1, this forces $\widetilde{\Gamma}'$ to be connected.

Let $\Gamma = (\mathsf{V}, \mathsf{E})$, and write $v = |\mathsf{V}|$, $e = |\mathsf{E}|$. Similarly, let $\widetilde{\Gamma}' = (\mathsf{V}', \mathsf{E}')$, and write $v' = |\mathsf{V}'|$, $e' = |\mathsf{E}'|$. We claim $v' = e' + 1$, and thus, $\widetilde{\Gamma}'$ is a tree.

Note that $v' = b_1(G_{\widetilde{\Gamma}'}) = b_1(N_\Gamma) = v - 1 \geq 5$. Moreover, $\binom{v'}{2} - e' = \operatorname{rank} \mathfrak{H}_2(G_{\widetilde{\Gamma}'}) = \operatorname{rank} \mathfrak{H}_2(N_\Gamma)$. We also know that $\operatorname{rank} \mathfrak{H}_2(N_\Gamma) = \operatorname{rank} \mathfrak{H}_2(G_\Gamma)$, by Lemma 6.2. Since $\operatorname{rank} \mathfrak{H}_2(G_\Gamma) = \binom{v}{2} - e$, we conclude that $e' = e - v + 1$. From Lemma 2.7(i), we know $2v - e = 3$; hence, $v' = e' + 1$, as claimed.

By the discussion from Example 8.2, the components of $\mathcal{R}_1(N_\Gamma) = \mathcal{R}_1(G_{\widetilde{\Gamma}'})$ must intersect transversely. This contradicts Lemma 9.3. □

9.3. *Arrangement groups*

Another widely studied class of groups are the fundamental groups of complements of complex hyperplane arrangements; see for instance [**27**] and references therein. Bestvina-Brady groups associated to simply-connected flag complexes share some common features with arrangement groups. Indeed, if G is a group in either class, then:

- G admits a finite presentation, with commutator relators only;
- G is 1-formal;
- $\mathcal{R}_1(G)$ is a union of linear subspaces.

There is a rather striking similarity between Bestvina-Brady groups associated to complete multipartite graphs and the fundamental groups of complements of "decomposable" arrangements. Indeed, if G is a group in either class, then the derived Lie algebra of $\operatorname{gr}(G)$ is isomorphic to the derived Lie algebra of a product of free groups, and similarly for the derived Chen Lie algebra. For Bestvina-Brady groups, this was noted in Remark 5.8, while for decomposable arrangement groups, this is proved (by completely different methods) in Theorems 2.4 and 6.2 from [**23**].

Even so, there are many finitely presented Bestvina-Brady groups which are not arrangement groups.

Proposition 9.5. *Let Γ be the 1-skeleton of an extra-special triangulation of D^2. Then the Bestvina-Brady group N_Γ is not isomorphic to any arrangement group.*

Proof. If G is an arrangement group, then any two components of $\mathcal{R}_1(G)$ intersect only at 0, see [**17**]. By Lemma 9.3, this cannot happen for N_Γ. □

Propositions 9.4 and 9.5 together yield Theorem 1.5 from the Introduction.

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