MTH3414 — FALL 2000

BUNDLES AND CHARACTERISTIC CLASSES

PROF. A. SUCIU

Homework 3

1. Let ξ be a principal G-bundle over a paracompact space B, with classifying map $f : B \to BG$. Let H be a closed subgroup of G, and $i : H \to G$ the inclusion. Show:
$$\xi \text{ reduces to } H \quad\Longleftrightarrow\quad \exists\, g : B \to BH \text{ such that } Bi \circ g \sim f.$$

2. Let G be a connected Lie group. Show that:
$$k_G(S^n) \cong \pi_{n-1}(G)\,.$$
Describe the isomorphism explicitly.

3. Give an example of an orientable k-plane bundle over S^n with zero Euler class but with no non-zero section. (*Hint:* Try $k = 3, n = 4$, and use the preceding exercise.)

4. Suppose $\mathbb{CP}^n$ is differentiably embedded in $\mathbb{R}^k$. Find a (good!) lower bound for k.

5. (a) If $\mathbb{RP}^n$ embeds in $\mathbb{R}^{n+1}$, then $n = 2^r - 1$.
   (b) If $\mathbb{RP}^n$ immerses in $\mathbb{R}^{n+1}$, then $n = 2^r - 1$ or $n = 2^r - 2$.
   (c) If $n = 2^r$, there is no immersion of $\mathbb{RP}^n$ in $\mathbb{R}^{2n-2}$, and no embedding in $\mathbb{R}^{2n-1}$.

6. (a) Let ξ be a complex vector bundle and ξ^* its dual. Show that $c_i(\xi^*) = (-1)^i c_i(\xi)$.
   (b) Let λ_1 and λ_2 be two complex line bundles. Show that $c_1(\lambda_1 \otimes \lambda_2) = c_1(\lambda_1) + c_1(\lambda_2)$.

7. Consider the Hopf fibration $\xi = (S^1 \to S^{2n+1} \to \mathbb{CP}^n)$. Identify the Euler class $e(\xi)$, write down the Gysin sequence of ξ, and use it to rederive the structure of $H^*(\mathbb{CP}^n;\mathbb{Z})$.

8. Let X_d be a nonsingular hypersurface in $\mathbb{CP}^3$ of degree d. Compute the Chern numbers $c_1^2(X_d)$ and $c_2(X_d)$.

9. Let $F(\mathbb{C}^n)$ be the manifold of complete flags in $\mathbb{C}^n$. Show that:
$$H^*(F(\mathbb{C}^n);\mathbb{Z}) \cong \mathbb{Z}[x_1,\dots,x_n] \Big/ \left(\prod_{i=1}^{n}(1+x_i) = 1\right).$$