MTH3414 — FALL 2000

BUNDLES AND CHARACTERISTIC CLASSES

PROF. A. SUCIU

Homework 2

1. Let ξ be a fiber bundle with structure group G and transition function $\{g_{i,j}\}_{i,j\in I}$, defined with respect to an open cover $\{U_i\}_{i\in I}$ of the base space. Prove:
$$\xi \text{ trivial} \Longleftrightarrow \exists\, r_i : U_i \to G \text{ with } g_{i,j}(b) = r_i(b)^{-1} r_j(b) \text{ for } b \in U_i \cap U_j.$$

2. Let ξ be a k-plane bundle with transition functions $\{\phi_{ij}\}$. Show:
   (a) If $k = 1$, then $\xi^{\otimes m} := \xi \otimes \cdots \otimes \xi$ (m factors) has transition functions $\{\phi_{ij}^m\}$.
   (b) The *determinant bundle* $\det \xi := \wedge^k \xi$ has transition functions $\{\det \phi_{ij}\}$.
   (c) $\det(\mathrm{T}^*(G_k(\mathbb{R}^n))) \cong (\det \gamma_k(\mathbb{R}^n))^{\otimes n}$.

3. For the canonical principal bundles $p : S^n \to \mathbb{RP}^n$ and $p : S^{2n+1} \to \mathbb{CP}^n$ determine an atlas and compute the transition functions.

4. Let ξ_n be the principal $\mathbb{Z}_n$-bundle $p_n : S^1 \to S^1$, where $p_n(z) = z^n$. Consider the associated $\mathbb{Z}_n$-bundle $\eta_n = \xi_n[S^1]$, where $\mathbb{Z}_n \subset S^1$ acts on S^1 by left-translation. Show that η_n is not trivial as a $\mathbb{Z}_n$-bundle, but it is trivial as a (principal) S^1-bundle.

5. Consider the $\mathbb{Z}_n$-action on S^3 given by $(z_1, z_2) \mapsto (\zeta z_1, \zeta z_2)$, where $\zeta = e^{2\pi i/n}$. Let L_n be the orbit space.
   (a) Define a principal S^1-bundle $L_n \to S^2$.
   (b) What is the clutching function of this bundle?
   (c) Show that $L_1 = S^3$ and $L_2 = \mathrm{SO}(3)$.