MTH3414 — FALL 2000

BUNDLES AND CHARACTERISTIC CLASSES

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Homework 1

1. Let ξ^k be a k-plane bundle over a compact base space. Show: $\exists \eta^n$ such that $\xi^k \oplus \eta^n \cong \epsilon^{k+n}$.

2. Let ξ^k be a k-plane bundle over an n-dimensional CW-complex, $k > n$. Show: ξ stably trivial $\implies$ ξ trivial, i.e.: if $\xi^k \oplus \epsilon^r \cong \epsilon^{k+r}$ then $\xi^k \cong \epsilon^k$.

3. Let M^n be a smooth submanifold of $S^{n+k}, n < k$. Show: $\mathrm{T}M$ is stably trivial $\iff$ $\nu_{_M}$ is trivial.

4. Let $G_k(R^n)$ be the Grassmannian of k-planes in $\mathbb{R}^n$.
   (a) Show: $\mathrm{T}G_k(\mathbb{R}^n) \oplus (\gamma^k(\mathbb{R}^n) \otimes \gamma^k(\mathbb{R}^n)) \cong n\gamma^k(\mathbb{R}^n)$
   (b) Find a classifying map for $\mathrm{T}\mathbb{RP}^n$. More generally, for $\mathrm{T}G_k(\mathbb{R}^n)$.

5. Fix a bijection $e : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and let $\mu : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ be defined by $\mu((x_i)_i, (y_j)_j) = ((x_i y_j)_{e(i,j)})$. Show:
   (a) μ induces a map $\mu : G_k(\mathbb{R}^\infty) \times G_l(\mathbb{R}^\infty) \to G_{kl}(\mathbb{R}^\infty)$ (called the Segre map).
   (b) If ξ, η have classifying maps f_ξ and f_η, then $\xi \times \eta$ has classifying map $\mu \circ (f_\xi \times f_\eta)$.

6. Let $\widetilde{G}_k(\mathbb{R}^n)$ be the Grasmannian manifold of *oriented* k-planes in $\mathbb{R}^n$.
   (a) There is a double covering $\mathbb{Z}_2 \to \widetilde{G}_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$.
   (b) There is a principal bundle $\mathrm{SO}(k) \times \mathrm{SO}(n-k) \to \mathrm{SO}(n) \to \widetilde{G}_k(\mathbb{R}^n)$.
   (c) Interpret the covering map $G_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$ from (a) as a map of homogeneous spaces, $\mathrm{SO}(n)/\mathrm{SO}(k) \times \mathrm{SO}(n-k) \to \mathrm{O}(n)/\mathrm{O}(k) \times \mathrm{O}(n-k)$.
   (d) Show that $\widetilde{G}_2(\mathbb{R}^4)$ is diffeomorphic to $S^2 \times S^2$.
   (e) What is the involution on $S^2 \times S^2$ that corresponds to the non-trivial covering map $\widetilde{G}_2(\mathbb{R}^4) \to \widetilde{G}_2(\mathbb{R}^4)$?

7. Let $p : E \to B$ be a real vector bundle of rank k. Let
   $$F(E) = \{(b, \mathbf{f}) \mid b \in B \text{ and } \mathbf{f} = (f_1, \dots, f_n) \text{ is a basis for } p^{-1}(b)\}$$
   be the *space of frames* of E, and let $q : F(E) \to B$ be given by $q(b, \mathbf{f}) = b$. Show that:
   (a) $q : F(E) \to B$ is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle.
   (b) The given vector bundle is associated to its frame bundle via the natural action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$, i.e., $E = F(E) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n$.