Northeastern University
Department of Mathematics

Prof. Alex Suciu **MTH 3107 – TOPOLOGY II** **Winter 1999**

Take-Home Final Exam

Due Monday, March 22

Instructions: *Do at least 5 of the following 6 problems. Give complete proofs or justifications for each statement you make. Show all your work.*

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1. Let X be a space. Show that:
   (a) $\widetilde{H}_n(X;\mathbb{Z}) = 0$ for all n if and only if $\widetilde{H}_n(X;\mathbb{Q}) = 0$ and $\widetilde{H}_n(X;\mathbb{Z}_p) = 0$ for all n and all primes p.
   (b) $f : X \to Y$ induces isomorphisms in $H_*(-;\mathbb{Z})$ if and only if it induces isomorphisms in $H_*(-;\mathbb{Q})$ and $H_*(-;\mathbb{Z}_p)$ for all primes p.

2. Prove the following theorem of Borsuk: If $f : S^n \to S^n$ commutes with the antipodal map, then f has odd degree.
   Remarks:
   (i) There is a direct (and rather long) proof in Bredon's book (Theorem 20.6, pp. 244-245). Use instead the Borsuk-Ulam theorem (see hint to Problem 7, p. 245).
   (ii) Note that this is *not* a homotopy-theoretic result. Indeed, $fa \simeq af$, for all f (show that!).

3. Let $X = K \times \mathbb{RP}^3$ be the product of the Klein bottle with the 3-dimensional projective space.
   (a) Find a CW-decomposition of X.
   (b) Determine the chain complex $(C_\bullet(X), d)$ associated to that cell decomposition.
   (c) Compute the (cellular) homology groups $H_*(X)$ and $H_*(X;\mathbb{Z}_2)$.

4. Recall that, for a space X, and a short exact sequence $0 \to G' \xrightarrow{\alpha} G \xrightarrow{\beta} G'' \to 0$ of abelian groups, there is an associated long exact in homology,
   $$\cdots \to H_i(X;G') \xrightarrow{\alpha_*} H_i(X;G) \xrightarrow{\beta_*} H_i(X;G'') \xrightarrow{\partial} H_{i-1}(G') \to \cdots$$
   Compute explicitly this homology sequence (the terms and the maps), in case the coefficients sequence is $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$, and
   (a) $X = \mathbb{RP}^3$.
   (b) $X = K$, the Klein bottle.

5. Let A be an integral 3×3 matrix, and let $f : T^3 \to T^3$ be the induced self-map of the 3-torus. Compute the Lefschetz number $L(f)$ in terms of the entries of A.

6. Let X be a finite simplicial complex, and let G be a finite group acting simplicially on X. Show that, for every $g \in G$,
   $$L(g) = \chi(X^g)$$
   where $X^g = \{x \in X \mid gx = x\}$.