Geometric interpretation of some linear transformations

1. Rotation-dilations

To interpret the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \vec{x}$$

geometrically, write the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in polar coordinates: $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$. Then the matrix of T can be written as

$$A = r \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Thus, T is counterclockwise rotation through the angle α, followed by a dilation by the factor r.

2. Projection-dilations

To interpret the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & b \\ b & b^2/a \end{bmatrix} \vec{x}$$

geometrically, write $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$. Then the matrix of T can be written as

$$A = \frac{r}{\cos\alpha} \begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix}.$$

Thus, T is orthogonal projection onto the line L making angle α with the x-axis, followed by a dilation by the factor $\dfrac{r}{\cos\alpha}$.

3. Reflection-dilations

To interpret the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \vec{x}$$

geometrically, write $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos 2\alpha \\ r\sin 2\alpha \end{bmatrix}$. Then the matrix of T can be written as

$$A = r \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}.$$

Thus, T is reflection in the line L making angle α with the x-axis, followed by a dilation by the factor r.