Prof. A. Suciu

MTH 1230 | **LINEAR ALGEBRA** | **Spring 1999**

MAKE-UP QUIZ 2

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1. $\boxed{\text{10 points}}$ Consider the matrix $A = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 4 & 1 \\ 4 & 3 & a \end{bmatrix}$.

(a) For which values of a is A invertible?

(b) Find the inverse of A when $a = 2$.

(c) Use the answer in part (b) to solve the system $A\vec{x} = \vec{b}$, where $a = 2$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

2. 8 points Let

$$A = \begin{bmatrix} 2 & -4 \\ 1 & 5 \\ 3 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix}.$$

Decide whether the following products are defined or not. If they are, compute them:

$$A \cdot B, \quad B \cdot A, \quad A \cdot C, \quad C \cdot A, \quad B \cdot C, \quad C \cdot B.$$

3. 4 points For three invertible $n \times n$ matrices A, B, and C, determine which of the following formulas are necessarily true:

(a) $(AB^3A^{-1})^{-1} = AB^{-3}A^{-1}$.

(b) $A(2B - 3C^{-1}) = 2AB - 3AC^{-1}$.

(c) $CBAA^{-1}C^{-1}B^{-1} = I_n$.

(d) $(A + B + C)^2 = A^2 + B^2 + C^2 + 2AB + 2AC + 2BC$.

4. 8 points Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(\vec{x}) = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \vec{x}.$$

(a) Sketch the image of the unit square under this transformation. (Indicate the coordinates of the vertices of the image, and shade the image.)

(b) Explain how T can be interpreted as a projection on a line L, followed by a dilation. What is the angle made by L with the x-axis, and what is the factor of dilation?

Geometric interpretation of some linear transformations

1. Rotation-dilations

To interpret the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \vec{x}$$

geometrically, write the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in polar coordinates: $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$. Then the matrix of T can be written as

$$A = r \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Thus, T is counterclockwise rotation through the angle α, followed by a dilation by the factor r.

2. Projection-dilations

To interpret the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & b \\ b & b^2/a \end{bmatrix} \vec{x}$$

geometrically, write $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$. Then the matrix of T can be written as

$$A = \frac{r}{\cos\alpha} \begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix}.$$

Thus, T is orthogonal projection onto the line L making angle α with the x-axis, followed by a dilation by the factor $\dfrac{r}{\cos\alpha}$.

3. Reflection-dilations

To interpret the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \vec{x}$$

geometrically, write $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos 2\alpha \\ r\sin 2\alpha \end{bmatrix}$. Then the matrix of T can be written as

$$A = r \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}.$$

Thus, T is reflection in the line L making angle α with the x-axis, followed by a dilation by the factor r.