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MTH 1230 **LINEAR ALGEBRA** **Spring 1999**

ANSWERS TO QUIZ 5

1. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$. We have: $\operatorname{rref} A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$, $\operatorname{rref} A^\top = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

(a) Find a basis for $\ker A$: $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

(b) Find a basis for $(\ker A)^\perp$: $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(c) Find a basis for $\ker A^\top$: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(d) Find a basis for $(\ker A^\top)^\perp$. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(e) Which one of the above four linear subspaces—$\ker A$, $(\ker A)^\perp$, $\ker A^\top$, $(\ker A^\top)^\perp$—equals $\operatorname{im} A$, and which one equals $\operatorname{im} A^\top$?

$$(\operatorname{im} A)^\perp = \ker A^\top \quad \Longrightarrow \quad \operatorname{im} A = (\ker A^\top)^\perp$$
$$(\operatorname{im} A^\top)^\perp = \ker A \quad \Longrightarrow \quad \operatorname{im} A^\top = (\ker A)^\perp$$

(f) What is the area of the parallelogram spanned by the column vectors of $5I_2 - A$?

$$|\det(5I_2 - A)| = \left|\det \begin{bmatrix} 4 & -3 \\ -2 & -1 \end{bmatrix}\right| = |-10| = 10$$

2. A company gathers the following data:

Year	1995	1996	1997	1998
Annual Sales (in millions of dollars)	2.0	2.5	3.2	4.1

Represent the years $1995, 1996, 1997, 1998$ as $0, 1, 2, 3$, respectively, and let x denote the year. Let y denote the annual sales (in millions of dollars).

(a) Find the least squares line relating x and y.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}; \qquad A^\top A = \begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix}; \qquad (A^\top A)^{-1} = \begin{bmatrix} 0.2 & -0.3 \\ -0.3 & 0.7 \end{bmatrix};$$

$$\begin{bmatrix} m \\ b \end{bmatrix} = (A^\top A)^{-1} A^\top \cdot y = \begin{bmatrix} -0.3 & -0.1 & 0.1 & 0.3 \\ 0.7 & 0.4 & 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} 2.0 \\ 2.5 \\ 3.2 \\ 4.1 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 1.9 \end{bmatrix}$$

The least square line has equation $y = mx + b$, that is, $y = 0.7x + 1.9$.

(b) Use the equation obtained in part (a) to estimate the annual sales for the year 2000.
The year 2000 corresponds to $x = 5$; the predicted sales are $y(5) = 0.7 \cdot 5 + 1.9 = 5.4$ million dollars.

3. Let A and B be two 3×3 matrices, with $\det A = -2$ and $\det B = 0$.

(a) Is A invertible? If yes, compute $\det(A^{-1})$. If not, say so.
Since $\det A \neq 0$, A is invertible, and $\det(A^{-1}) = (\det A)^{-1} = -\frac{1}{2}$.

(b) Is B invertible? If yes, compute $\det(B^{-1})$. If not, say so.
Since $\det B = 0$, B is not invertible.

(c) Compute: $\det(4A) = 4^3 \det A = -128$.

(d) Compute: $\det(A^4) = (\det A)^4 = 16$.