Prof. Alexandru Suciu

MTH 1230 **LINEAR ALGEBRA** **Spring 2001**

SOLUTIONS to EXAM 1

1. $\boxed{\text{10 points}}$

(a) Use Gauss-Jordan elimination (rref) to solve the following system of equations. Carry out the elimination all the way, before solving.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + \ \ 4x_4 &= \ 5 \\ 2x_1 + 5x_2 + 7x_3 + 11x_4 &= 12 \\ x_2 + \ \ x_3 + \ \ 4x_4 &= \ 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \vdots & 5 \\ 2 & 5 & 7 & 11 & \vdots & 12 \\ 0 & 1 & 1 & 4 & \vdots & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 & 0 & \vdots & 3 \\ 0 & 1 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

(b) Explain why there are no solutions, an infinite number of solutions (write down the general solution in vector form if this is the case), or exactly one solution (write down the solution in this case).

The coefficient matrix has fewer pivot columns (3) than columns (4), and no row of 0's. Thus, the system has infinitely many solutions, depending on $4 - 3 = 1$ parameters. Set x_3 (the free variable) equal to t (the parameter), and solve for the other (leading) variables:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

2. $\boxed{\text{9 points}}$ Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Use Gauss-Jordan elimination to compute the inverse of the matrix A.

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

(b) Use the answer in part (a) to solve the system $A\vec{x} = \vec{b}$.

$$\vec{x} = A^{-1} \cdot \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

3. $\boxed{\text{6 points}}$ The reduced row echelon forms of the augmented matrices of 3 systems are given below. In each case, indicate the rank of the matrix of coefficients (to the left of the dotted lines), and the number of solutions of the system (you need **not** write down the solutions.)

(a) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$ rank $= 4$, # solutions $= 1$

(b) $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ rank $= 2$, # solutions $= 0$

(c) $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ rank $= 2$, # solutions $= \infty$

4. $\boxed{\text{3 points}}$ Find the matrix A of the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ given by

$$\begin{aligned} y_1 &= x_2 - 2x_3 + x_4 \\ y_2 &= -x_1 + x_3 + x_4 - x_5 \\ y_3 &= 2x_1 - x_2 + 3x_5 \end{aligned} \qquad A = \begin{bmatrix} 0 & 1 & -2 & 1 & 0 \\ -1 & 0 & 1 & 1 & -1 \\ 2 & -1 & 0 & 0 & 3 \end{bmatrix}$$

5. $\boxed{\text{3 points}}$ Write down (in format as above) the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}, \quad T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-5\\6\end{bmatrix}, \quad T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}. \qquad \begin{aligned} y_1 &= 3x_1 - 5x_2 + x_3 \\ y_2 &= 2x_1 + 6x_2 + 4x_3 \end{aligned}$$

6. $\boxed{\text{6 points}}$ Find the matrix A of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$T\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}, \quad T\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-3\\4\end{bmatrix}.$$

$$A \cdot \begin{bmatrix}3 & 2\\2 & 1\end{bmatrix} = \begin{bmatrix}2 & -3\\-1 & 4\end{bmatrix} \implies A = \begin{bmatrix}2 & -3\\-1 & 4\end{bmatrix} \cdot \begin{bmatrix}3 & 2\\2 & 1\end{bmatrix}^{-1} = \begin{bmatrix}2 & -3\\-1 & 4\end{bmatrix} \cdot \begin{bmatrix}-1 & 2\\2 & -3\end{bmatrix} = \begin{bmatrix}-8 & 13\\9 & -14\end{bmatrix}$$

7. $\boxed{\text{4 points}}$ Sketch the image of the unit square under the linear transformation $T(\vec{x}) = \begin{bmatrix}1 & 3\\-2 & 2\end{bmatrix}\vec{x}$.

This is the parallelogram spanned by $T(\vec{e}_1) = \begin{bmatrix}1\\-2\end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix}3\\2\end{bmatrix}$.

8. $\boxed{\text{9 points}}$ Find the matrices of the following linear transformations:

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$, a counter-clockwise rotation of 60°, followed by a dilation by a factor of 3.

$$A = \begin{bmatrix} \frac{3}{2} & -\frac{3\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} & \frac{3}{2} \end{bmatrix}$$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$, the reflection in the y–z-plane.

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) $T : \mathbb{R}^3 \to \mathbb{R}^2$, the projection onto the y–z-plane.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$