1	2	3	4	5	6	7	8	Σ

Name: ______________________

NORTHEASTERN UNIVERSITY
DEPARTMENT OF MATHEMATICS

MTH 1230 — FINAL EXAM — **Spring 2001**

Instructions: Put your name in the blanks above. Put your final answers to each question in the designated spaces. Calculators are permitted. A single sheet of formulas is allowed. **Show your work.** If there is not enough room to show your work, use the back page.

1. [12 points] Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix}.$$

(a) Are the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$ linearly independent or dependent? If they are independent, say why. If they are dependent, exhibit a linear dependence relation among them.

(b) Write the vector $\vec{b} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ as a linear combination of the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$.

2. 12 points The matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{bmatrix}$ has the matrix $E = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ as its row-reduced echelon form.

(a) Find a basis for the image of A.

(b) Find a basis for the kernel of A.

(c) Compute:

- $\text{rank}\, A =$
- $\dim(\text{im}\, A) =$
- $\dim(\ker A) =$
- $\dim(\text{im}\, A)^{\perp} =$
- $\dim(\ker A)^{\perp} =$

3. [12 pts] The number of students getting an A on the Spring final exam of a certain Linear Algebra course is as follows:

Year	1997	1998	1999	2000
A's	2	1	4	6

Represent the years $1997, 1998, 1999, 2000$ as $0, 1, 2, 3$, respectively, and let t denote the year (after 1997). Let y denote the number of A's.

(a) Find the line $y = mt + b$ that best fits the above data points, using the least squares method.

(b) Use the equation obtained in part (a) to estimate how many students will get an A in Linear Algebra in Spring 2001.

4. 14 pts Consider the independent vectors

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) Use the Gram-Schmidt process to find an othonormal basis, $\vec{w}_1,\ \vec{w}_2,\ \vec{w}_3$, for the space spanned by the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$.

(b) Compute the volume of the parallelepiped spanned by the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$.

5. 14 pts Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates the xz-plane by $120°$ and reflects the y-axis about the xz-plane.

(a) Find the matrix A corresponding to T.

(b) What is $\det(A)$?

(c) Is A orthogonal? Why, or why not?

(d) Find A^{-1}.

(e) What is the image of the vector $\begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$ under the map T?

6. [14 points] A 4×4 matrix A has eigenvalues $\lambda_1 = -2$, $\lambda_2 = 1$, $\lambda_3 = 3$, $\lambda_4 = 4$.

(a) What is the characteristic polynomial of A?

(b) Compute $\operatorname{tr}(A)$ and $\det(A)$.

(c) Compute $\det(-2A)$.

(d) Compute $\det(A + 2I_4)$.

(e) What are the eigenvalues of A^3?

(f) Compute $\operatorname{tr}(A^3)$ and $\det(A^3)$.

(g) Is A invertible? Why, or why not?

(h) Is A orthogonal? Why, or why not?

(i) Is A diagonalizable? Why, or why not?

7. 10 points Find a 2×2 matrix A such that $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors of A, with eigenvalues -2 and 5, respectively.

8. 14 pts Let $A = \begin{bmatrix} 5 & 6 & 0 \\ 7 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues of A.

(c) Find a basis for each eigenspace of A.

(d) Find an invertible matrix S and a diagonal matrix D such that $A = S \cdot D \cdot S^{-1}$. [You do not have to calculate S^{-1}.]

9. **Extra Credit: 10 pts** Let $A = \begin{bmatrix} 27 & -12 \\ 56 & -25 \end{bmatrix}$. Write A^t (the matrix A raised to the power t, a positive integer) in the form of a single 2×2 matrix. (You may use the fact that the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector with associated eigenvalue 3, and the vector $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ is an eigenvector with associated eigenvalue -1.)