Prof. Alexandru Suciu

MTH 1125 **Calculus 3** **Spring 2002**

SOLUTIONS to QUIZ 3

1. [4 points] You are given a formula for the nth term a_n of a sequence $\{a_n\}$. Find the values a_1, a_2, a_3, a_4.

$$a_n = \frac{(-1)^{n+1}}{n^2+1}$$

$$a_1 = \frac{1}{2}, \qquad a_2 = -\frac{1}{5}, \qquad a_3 = \frac{1}{10}, \qquad a_4 = -\frac{1}{17}$$

2. [4 points] Does the following sequence converge, or does it diverge? Find the limit if it is a convergent sequence. **Explain!**

$$a_n = \frac{\cos n}{n^2}$$

Recall that

$$-1 \leq \cos(n) \leq 1$$

Hence

$$\frac{-1}{n^2} \leq \frac{\cos n}{n^2} \leq \frac{1}{n^2}$$

Now, since

$$\lim_{n\to\infty} \frac{-1}{n^2} = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

we conclude, by the Sandwich Principle, that the sequence $\{a_n\}$ **converges** to the same limit:

$$\lim_{n\to\infty} a_n = 0$$

3. 4 points Does the following sequence converge, or does it diverge? Find the limit if it is a convergent sequence. **Explain!**

$$a_n = \frac{7 + 10n^4}{4n^3 - n^2 + 5}$$

We can compute the limit two ways, either by dividing top and bottom by n^4:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{7}{n^4} + 10}{\frac{4}{n} - \frac{1}{n^2} + \frac{5}{n^4}} = \frac{0 + 10}{0 - 0 + 0} = \infty$$

or by applying repeatedly l'Hôpital's rule:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{40n^3}{12n^2 - 2n} = \lim_{n\to\infty} \frac{120n^2}{24n - 2} = \lim_{n\to\infty} \frac{240n}{24} = \infty$$

Either way, we find that the sequence **diverges**.

4. 4 points Show that the following sequence is decreasing.

$$a_n = \frac{4n + 3}{2n + 1}$$

- **First method.** We verify directly that $a_n > a_{n+1}$, for all n:

$$\frac{4n+3}{2n+1} > \frac{4n+7}{2n+3}$$
$$8n^2 + 12n + 6n + 9 > 8n^2 + 4n + 14n + 7$$
$$9 > 7 \quad \text{checks!}$$

- **Second method.** Let $f(x) = \dfrac{4x+3}{2x+1}$. Then:

$$f'(x) = \frac{4(2x+1) - 2(4x+3)}{(2x+1)^2} = -\frac{2}{(2x+1)^2} < 0$$

  and so $f(x)$ is decreasing. Thus, the sequence with terms $a_n = f(n)$ is also decreasing.

5. 4 points A sequence $\{a_n\}$ is given recursively, as follows:

$$a_1 = \sin(1) \qquad a_2 = \max\{a_1, \sin(2)\} \qquad \dots \qquad a_{n+1} = \max\{a_n, \sin(n+1)\} \qquad \dots$$

Show that the sequence converges.

- The sequence is increasing (to be more precise, non-decreasing), since a_{n+1} is at least as big as a_n, by definition.
- The sequence is bounded above by the constant $M = 1$, since

$$a_n \le \max\{\sin(1), \dots, \sin(n)\} \le 1$$

- Thus, by the Theorem on monotone, bounded sequences, our sequence $\{a_n\}$ **converges**.