Lecture 6 — Yuzvinsky

L, $D = D(L)$ $\quad D = \langle \sigma \subset \mathcal{A},\ \deg\sigma = |\sigma| \rangle_k$

$\mathcal{A}$ = atoms

$$d(\sigma) = \sum_{\substack{j=1 \\ \vee\sigma_j = \vee\sigma}}^{\deg\sigma} (-1)^{j-1}\sigma_j \qquad \tilde{H}_p(L) = H_{p+2}(D)$$

Folkman Thm $L = \cap$-lattice

$$H_p(D_X) = \begin{cases} 0 & p \neq \operatorname{codim} X \\ |\mu(X)| & p = \operatorname{codim} X \end{cases}$$

P = poset

$$\mu = \delta^{-1},\ \mu_{X,Y} = \mu(X,Y) \qquad \delta = \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix}$$

$$\mu(X) = \mu(V, X)$$

$$\delta_{XY} = \begin{cases} 1 & X \leq Y \\ 0 & \text{otherwise} \end{cases}$$

Pr 1: Prove the 1st part of F.T.
(Hint: compute $\operatorname{codim} X - 2$-skeleton of atomic complex of L_X)

Convert D to a diff. alg.

$$\sigma \cdot \tau = \begin{cases} 0 & \text{if } \operatorname{codim} \vee\sigma\cup\tau \neq \operatorname{codim}\vee\sigma + \operatorname{codim}\vee\tau \\ (\sigma\cup\tau)\,\varepsilon(\sigma,\tau) & \text{otherwise} \end{cases}$$

where $\varepsilon(\sigma,\tau)$ = sign of the shuffle of σ and τ

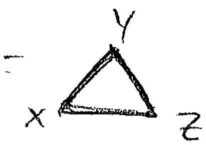
Pr 2: Check that $\bullet$ generates a DGA.

Note: $\forall i$ $\underset{e_i}{H_i} \in \mathcal{A}$ is a 1-cycle

Thm: The assignment $e_i \mapsto \xi_i$ generates a graded K-algebra iso. $A \xrightarrow{\sim} H_*(D)$

Cor. $$H(A,t) = \sum_{X\in L} |\mu(X)|\, t^{\operatorname{codim} X} = \mathrm{Poin}(L,t)$$

Ex: $\overset{1}{x}\ \overset{2}{y}\ \overset{3}{z}\ \overset{4}{x-y}\ \overset{5}{x-z}\ \overset{6}{y-z}$

$K =$ 

$$H(A,t) = 1+6t+11t^2+6t^3$$

$$S.R.(K) = k[x,y,z]/(xyz)$$

x^2, y^2, z

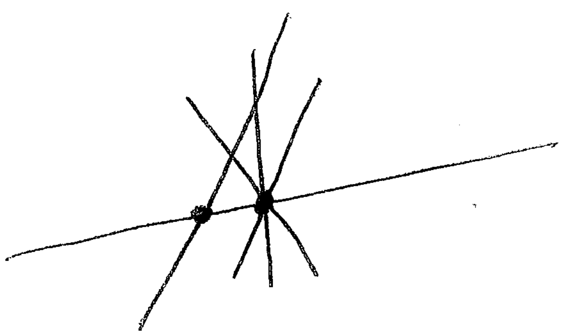
Deletion-Restriction Exact Seq.

$\mathcal{A}$, $\mathcal{A}' = \mathcal{A}\setminus\{H_n\}$, $\mathcal{A}'' = \{H_n \cap H \mid H \in \mathcal{A}\}$

$H_n \in \mathcal{A}$

Have OS.'s A, A', A''

Ex



H_n

There is the exact seq. of graded K-modules

$$0 \to A' \xrightarrow{i} A \xrightarrow{j} A''(-1) \to 0$$

Pr 3: i is an embedding of algebras.

$$j(e_S) = \begin{cases} e_{\lambda(S\setminus\{n\})} & \text{if } n\in S \\ 0 & \text{if } n\notin S \end{cases}$$

where $\lambda(H_i) = H_i \cap H_n$, $i \neq n$

(Pr 4)* Prove exactness

Topological interpretation

Thm: "$A(\mathcal{A}) \simeq H^*(M(\mathcal{A}))$"
as graded k-alg.
(Arnold, Brieskorn, O-S)

Have exact sequence

$$\begin{array}{ccccccccc} 0 & \to & A' & \to & A & \to & A'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H^*(M') & \to & H^*(M) & \to & H^{*-1}(M'') & \to & 0 \end{array}$$

$M \subset M'$, $M'' \subset M'$ $\quad M' \setminus M'' = M$

N = Tubular neighborhood of M'' in M'

$N \supset \mathbb{C}$ trivial, $N \downarrow M''$ $\quad$ So, $N = M'' \times \mathbb{C}$, $N_0 = N \setminus M''$

So, $(N, N_0) = M'' \times (\mathbb{C}, \mathbb{C}^*)$

and $H^*(N, N_0) = H^*(M'') \otimes H^2(\mathbb{C}, \mathbb{C}^*)$, $\quad H^2(\mathbb{C}, \mathbb{C}^*) \cong H^1(\mathbb{C}^*) \simeq \mathbb{Z} =: \langle t \rangle$

$H^*(N, N_0) \underset{+2}{\overset{t}{\xleftarrow{\ \simeq\ }}} H^*(M'')$

Exact seq. of (M', M)

$$\cdots \to H^p(M') \xrightarrow{i^*} H^p(M) \xrightarrow{\delta} H^{p+1}(M', N) \xrightarrow{j^*} H^{p+1}(M') \to \cdots$$

Excise $M' \setminus N$ from (M', M)

iso. $\alpha : H^*(M', M) \xrightarrow{\sim} H^*(N, N_0)$

$$\cong H^*(M'')$$

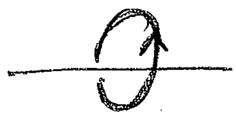
diff. form

$$\frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z}$$

$\mathbb{C}$

Pull back $\frac{dz}{z}$

$$\boxed{\omega_K = \frac{d\alpha_K}{\alpha_K}} \longleftrightarrow$$



$[\omega_K] \in H^1(M)$

Pr 5: Prove that $\left(\frac{d\alpha_K}{\alpha_K}\right)$ satisfy all the relations on e_i in A

This gives us the maps

$$e_i \longmapsto \omega_i$$

$$A \longrightarrow H^*(M).$$

Thm: Assignment $e_i \longmapsto \omega_i$ generates a graded alg. iso

$$A \xrightarrow{\sim} H^*(M)$$

Cor.: $P(M,t) = H(A,t) = \mathrm{Poin}(L,t)$

$$= \sum_{X \in L} |\mu(X)|\, t^{\operatorname{codim} X}$$