Lecture 4

Yuzvinsky

$\boxed{xy = -yx}$

$$\frac{k\langle e_1, \dots, e_n\rangle}{e_ie_j + e_je_i}$$

1) $\mathcal{A} = \{H_1, \dots, H_n\}$, $H_j \overset{\text{codim } 1}{\subset} \mathbb{C}^\ell$

Fix K = a comm. ring

Let E be the exterior alg. over K with generators $e_1, \dots, e_n$; $\deg e_i = 1$

$E = \bigoplus_{i=0}^{n} E_i$ (as free K-mod) $\deg E_i = i$

$E_0 = K$, $E_1 = \bigoplus_{j=1}^{n} K e_j$, $E_p = \bigwedge^p E_1$, $e_ie_j = -e_je_i$

$e_i^2 = 0$

$E_p = \bigoplus_{|S|=p} K \underbrace{e_{i_1} \cdots e_{i_p}}_{e_S}$ $\quad i_1 < \cdots < i_p \quad S = \{i_1, \dots, i_p\}$

Define $d : E \to E$, K-linear, $d^2 = 0$, $\deg d = -1$

$d : E_p \to E_{p-1}$ $\quad d(ab) = d(a)b + (-1)^{\deg a} a\, d(b)$

if a is homogeneous.

$de_i = 1, \forall i$.

It implies $de_S = \sum (-1)^{j-1} e_{S_j}$ $\quad S_j = S \setminus \{i_j\}$

$\underset{\text{ideal of } E}{I(\mathcal{A})} = (de_S \mid S = \underbrace{\text{minimal dependent set}}_{\text{circuits}})$

Ex: $\overset{1}{x}, \overset{2}{y}, \overset{3}{z}, \overset{4}{x-y}, \overset{5}{x-z}, \overset{6}{y-z}$

$e_1, \dots, e_6$

Dep. sets: 124, 135, 236, 456, 1256, ...

got as a generator of I

$$e_1e_2 - e_1e_4 + e_2e_4 = (e_1-e_2)(e_1-e_4)$$

$$e_1e_3 - e_1e_5 + e_3e_5$$

$$e_2e_3 - e_2e_6 + e_3e_6$$

$$e_4e_5 - e_4e_6 + e_5e_6$$

Pr 1: Show that these 4 generate $I(\mathcal{A})$.

$$\boxed{A = A(\mathcal{A}) = E/I(\mathcal{A})}$$

O.S.-alg. (Orlik-Solomon)

P2: If S is dep. then $e_S \in I$

2) **Dependence of Combinatorics**

$A(\mathcal{A})$ is defined by the matroid of $\mathcal{A}$.
= intersection lattice of $\mathcal{A}$ (= the set of all $\cap$'s of H_i's ordered opposite to inclusion).

Dimensions:

	0	1	2	3	4	5	6
E	1	6	15	20	15	6	1
A	1	6	11	6	0	0	0

$$H(\mathcal{A},t) = \sum_{p=0}^{\ell} \dim(A_p)\, t^p = 1 + 6t + 11t^2 + 6t^3$$

$$= (1+t)(1+2t)(1+3t)$$

The lattice

3
2
1
0

$\mathrm{rk}\, X = \mathrm{codim}\, X$
$X \in L$

A set S, of atoms is dependent iff

~~dep~~ $\operatorname{rk} V(S) < |S|$

a) A is graded by L, $\forall x \in L$

$$A_x = \langle e_S \mid VS = x \rangle_K$$

Pr 3 Prove A_x is well-defined.

$$A = \bigoplus_{\substack{k\text{-mod} \\ x \in L}} A_x \;;\quad A_p = \bigoplus_{\operatorname{rk} x = p} A_x$$

b) A is filtered by L. $A(X) = A(\delta\mathcal{L}_x)$

where $\delta\mathcal{L}_x = \{H \in \delta\mathcal{L} \mid X \subseteq H\}$ $\quad A \supset A(X)$

Pr 4 Prove $A \supset A(X)$.

Filtration is defined by the grading

$$A(X) = \bigoplus_{Y \leq X} A_Y$$

c) Homological Interpretation.

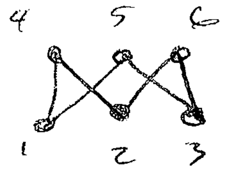
P = poset (finite)

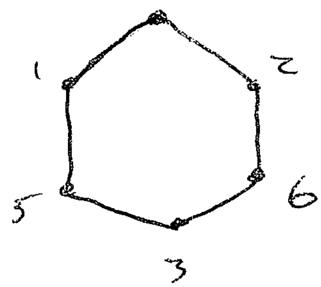
The ~~flag~~ (order) complex:

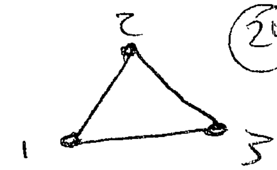
simplices ~~are~~ flags in P

= linearly ordered subset

Ex P







Via this, P gets alg. top.!

There are other complexes for P homotopy eq. to this.

Eg. If P is a lattice it is atomic complex (with highest and lowest pts deleted)

= simplices are sets of atoms bounded from above but not by highest element.

PrS: Prove homotopy equivalence of flag and atomic complexes for a lattice.

We need the relative atomic complex

~~(Δ simplex on atoms, atomic complex) or~~ the chain complex D

where $D_p = \langle \sigma \subseteq A, |\sigma| = p \rangle_K$

and $d : D_p \longrightarrow D_{p-1}$

$$d(\sigma) = \sum (-1)^{j-1} \sigma_j \qquad \sigma_j = \sigma \setminus \{H_{i_j}\} \qquad \sigma = \{H_{i_1}, \dots, H_{i_p}\}$$

$d^2 = 0$ $\qquad \vee\sigma_j = \vee\sigma$

① $\underset{\substack{\text{as}\\ \text{chain}\\ \text{complex}}}{D} = \bigoplus_{x \in L} D_x$

where $D_x = \langle \sigma \mid V(\sigma) = x \rangle$

② $D_x = \left(\Delta_x, \ \text{atomic complex} \right)$

more precisely $\forall x$ we have exact:

$$0 \to D'(x) \to D''(x) \longrightarrow \underset{\uparrow \text{sub. } x}{D(x)} \longrightarrow 0$$

$D'(x) = \text{atomic comp. on } A_x$ $\qquad D''(x) = \text{simplex on } A_x$

$$\Rightarrow \boxed{H_p(D_x) = \tilde{H}_{p-2}(L_x)} \qquad L_x = \{ y \in L \mid y \leq x \}$$

③ <u>Folkmann theorem</u> (1966)

For Γ-lattice

$$H_p(D_x) = \begin{cases} 0 & \text{codim } x \neq p \\ K^{m_x} & \text{codim } x = p \end{cases}$$

where $m_x = \mu_L(x)$