Overview of Aomoto (Gelfand) Theory of Hypergeometric Integrals (HG)

① Gauss HG diff. eq.

$$x(1-x)y'' + [c-(1+a+b)x]y' - aby = 0$$

has a solution

$$F[a,b,c,x] = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots$$

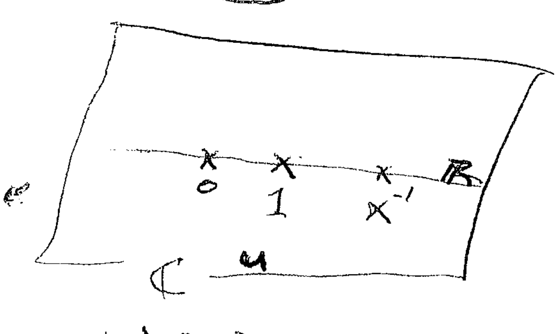
HG series

when $|x|<1$

$$\frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}\,du$$

$(\mathrm{Re}\, a>0,\ \mathrm{Re}(c-a)>0\ \ |x|<1)$

② 1-arrangement



$x \notin \{0,1\}$

parametrized 1-arr.'s

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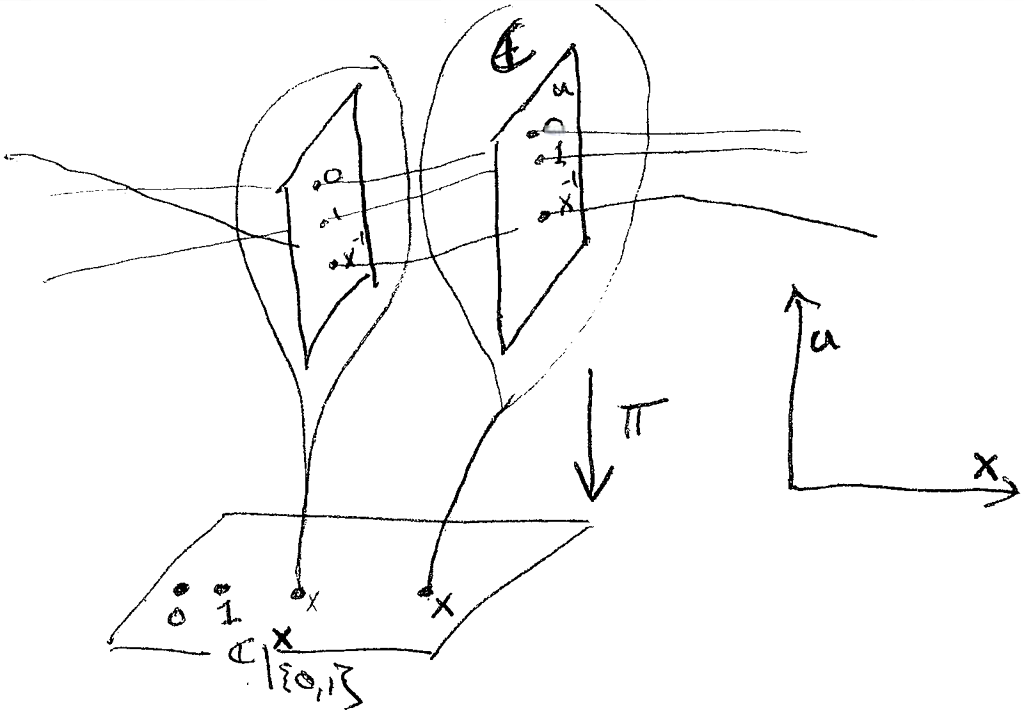
family of 1-arr.'s

combinatorially equiv.



$$\pi^{-1}(x) = M_x = \mathbb{C}\setminus\{0,1,x^{-1}\}$$

$$\Phi(u,\lambda,x) = (1-u)^{\lambda_1}\, u^{\lambda_2}\,(1-xu)^{\lambda_3} \qquad \lambda_i \in \mathbb{C}$$

is a multivalued function

$$\omega_\lambda = d\log \Phi(u,\lambda,x) = -\lambda_1 \frac{du}{1-u} + \lambda_2 \frac{du}{u} - \lambda_3 \frac{x\,du}{1-x}$$

holomorphic funcs sheaf — holo-1-forms

$$\nabla_x : \mathcal{O}_{M_x} \xrightarrow{\mathbb{C}\text{-linear}} \Omega^1_{M_x}$$
$$f \longmapsto df + f\omega_\lambda$$

$$\operatorname{Ker} \nabla_x = \{ f \in \mathcal{O}_{M_x} \mid df = -f\omega_\lambda \} \overset{\star}{=} \mathbb{C}\left(\frac{1}{\Phi}\right)$$

$$\nabla_x(\Phi^{-1}) = d\left(\frac{1}{\Phi}\right) + \Phi^{-1}\omega_\lambda = -\frac{1}{\Phi^2}\,d\Phi + \frac{1}{\Phi}\,d\log\Phi =$$

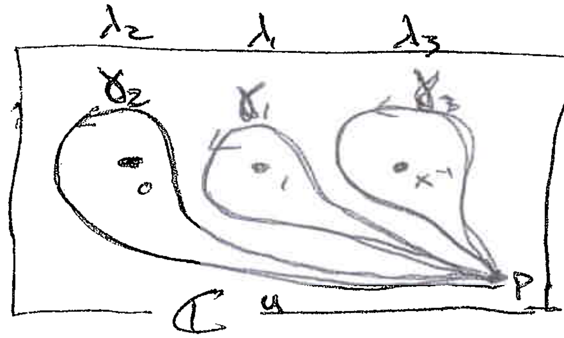
$\mathcal{L}_x := \operatorname{Ker} \nabla_x$ locally constant sheaf

$\mathcal{L}_x$ determines a representation

local system

$$\rho : \pi_1(M_x, p) \longrightarrow \mathrm{Aut}(\mathbb{C}) = \mathbb{C}^*$$

$$\gamma_j \longmapsto \exp(-2\pi i \lambda_j)$$



Consider $\lambda = 0$, then $\Phi = 1$ and $\omega_\lambda = 0$ and

$\nabla = d.$ $\quad \mathcal{L}_x = \mathbb{C}$

<u>deRham pairing</u> (nontwisted) non-degenerate pairing

$$H^1(M_x; \mathbb{C}) \times H_1(M_x; \mathbb{C}) \to \mathbb{C}$$

$$[\omega] \qquad [\sigma] \longmapsto \int_\sigma \omega$$

(Stokes) $\displaystyle\int_{\partial\sigma} \omega = \int_\sigma d\omega$

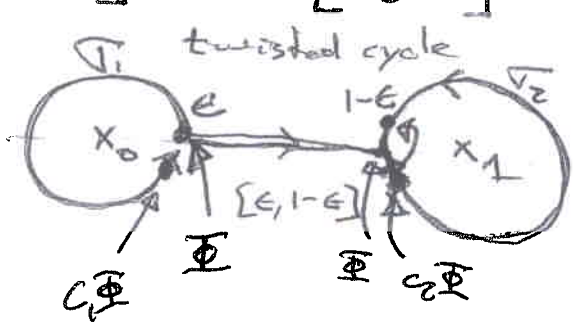
<u>Twisted deRham pairing</u>

$$H^1(M_x; \mathcal{L}_x) \times H_1(M_x; \mathcal{L}_x^\vee) \xrightarrow{\text{non-deg.}} \mathbb{C}$$

dual local system

$$[\omega] \qquad [\sigma \otimes \Phi] \qquad \int_\sigma \Phi\omega$$

twisted cycle

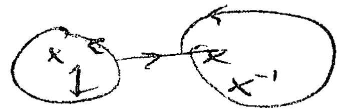


Sps $c_1 \neq 1$ & $c_2 \neq 1$

$$\sigma = \frac{1}{c_1 - 1}\sigma_1 \otimes \Phi$$
$$+ [1-\epsilon, \epsilon] \otimes \Phi$$
$$- \frac{1}{c_2-1}\sigma_2 \otimes \Phi$$

So, $\partial\sigma = 0$

$$\partial\sigma = \frac{1}{c_1-1}\left(c_1[\epsilon]\otimes\Phi - [\epsilon]\otimes\Phi\right) + \left([1-\epsilon]\otimes\Phi - [\epsilon]\otimes\Phi\right) - \frac{1}{c_2-1}\left(c_2[1-\epsilon]\otimes\Phi - [1-\epsilon]\otimes\Phi\right)$$

The other cycle is 

$$\int_{\partial\sigma}\Phi\,\omega = \int_{\sigma} d(\Phi\,\omega) = \int_{\sigma}\Phi\underbrace{\left(d\log\Phi\wedge\omega + d\omega\right)}_{\nabla_u\,\omega}$$

If λ is generic enough
($e.g.$ $\lambda_1\notin\mathbb{Z}$, $\lambda_2\notin\mathbb{Z}$, $\lambda_3\notin\mathbb{Z}$, and $\lambda_1+\lambda_2+\lambda_3\notin\mathbb{Z}$)

$$\dim H^1(M_x, \mathcal{L}_x) = 2$$

Also,

$$A_1 \curvearrowright \left(\mathbb{C}\frac{du}{1-u}\oplus\mathbb{C}\frac{du}{u}\oplus\mathbb{C}\frac{x\,du}{1-xu}\right)\Big/\nabla 1$$

basis for $H^1(M_x;\mathcal{L}_x)$ is $\begin{cases}\varphi_1 = \lambda_2\dfrac{du}{u}\\ \varphi_2 = -\lambda_3\dfrac{x\,du}{1-xu}\end{cases}$

twisted de Rham

$$0\to\Gamma(M_x,\mathcal{O})\xrightarrow{\nabla}\Gamma(M_x,\Omega^1)\xrightarrow{\nabla}\cdots$$

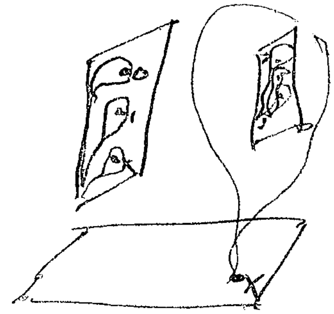
$$H^q(\Gamma(M_x,\Omega^*),\nabla)\cong H^q(M_x,\mathcal{L}_x)$$

$\hat{\varphi}_i = \int_\sigma \Phi \varphi_i \qquad [\sigma] \in H_1(M, \mathcal{L}_x^\vee)$

what kind of diff. equ's are satisfied by $\hat{\varphi}_1, \hat{\varphi}_2$?

diff. equ's $\longleftrightarrow$ cohomologous relations in $H^1(M_x; \mathcal{L}_x)$

d' exterior diff. w.r.t. x



$$d'\hat{\varphi}_1 = d'\int u^{\lambda_1}(1-u)^{\lambda_2}(1-ux)^{\lambda_3}\lambda_2\frac{du}{u}$$

$$= -\int u^{\lambda_1}(1-u)^{\lambda_2}(1-ux)^{\lambda_3}\, u\lambda_3\lambda_2\frac{du}{u}$$

$$= -\frac{\lambda_2\cancel{\lambda_3}}{u}\int u^{\lambda_1}(1-u)^{\lambda_2}(1-ux)^{\lambda_3}\underbrace{\frac{\cancel{\lambda_3}x\,du}{1-xu}}_{= -\varphi_2}dx$$

$$= \frac{\lambda_2\cancel{\lambda_3}}{x}\hat{\varphi}_2\,dx$$

So, $d'\hat{\varphi}_1 = \frac{\lambda_2}{x}\hat{\varphi}_2$

$$d'\begin{bmatrix}\hat{\varphi}_1\\ \hat{\varphi}_2\end{bmatrix} = \begin{bmatrix}0 & 0\\ \lambda_3 & \lambda_1+\lambda_3\end{bmatrix}\begin{pmatrix}\hat{\varphi}_1\\ \hat{\varphi}_2\end{pmatrix}\frac{dx}{x-1}$$

$$+\begin{bmatrix}0 & \lambda_2\\ 0 & -\lambda_1-\lambda_2\end{bmatrix}\begin{bmatrix}\hat{\varphi}_1\\ \hat{\varphi}_2\end{bmatrix}\frac{dx}{x}$$

To interpret this take

$$\Omega = \begin{pmatrix} 0 & 0 \\ \lambda_3 & \lambda_1+\lambda_3 \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} 0 & \lambda_2 \\ 0 & -\lambda_1-\lambda_2 \end{pmatrix} \frac{dx}{x}$$

log. forms

$$\nabla_\Omega := d' + \Omega \wedge$$

$\begin{bmatrix} \hat{\Phi}_1 \\ \hat{\Phi}_2 \end{bmatrix}$ is a solution for $\nabla_\Omega \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = 0$

Gauss-Manin connection

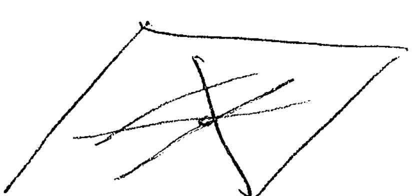
$\frac{d}{dx}\hat{\Phi}_1 = \frac{\lambda_2}{x}\hat{\Phi}_2$ Eliminate $\hat{\Phi}_2$ to have

$$x(1-x)y'' + [(\lambda_1+\lambda_2+1) - (\lambda_2-\lambda_3+1)x]y' + \lambda_2\lambda_3 y = 0$$

has solution $\hat{\Phi}_1$

Set $\begin{cases} \lambda_1 = c-a-1 \\ \lambda_2 = a \\ \lambda_3 = -b \end{cases}$ to recover Gauss H.G. diff. eqn.

③ <u>Higher</u> <u>Dimensional</u>



parametrized combinatorially equ.