Lecture 7 Terao

Free Arrangements

V is ℓ-dim v.s. over K

$\mathcal{A}$ a central arr. in V

$S = S(V^*)$, $Q = Q(\mathcal{A})$

1. Free Arr.

Def. 1.1 $\mathrm{Der}_S := \{\theta : S \to S \mid \theta$ is K-linear derivati[on]$\}$

Put $\partial_i(x_j) = \delta_{ij}$ $\mathrm{Der}_S \cong S^\ell$

So, $\forall \theta \in \mathrm{Der}_S$, $\theta = f_1\partial_1 + \cdots + f_\ell \partial_\ell$

Def 1.2 $D(\mathcal{A}) := \{\theta \in \mathrm{Der}_S \mid \theta(Q) \in QS\}$

Pr 1: Prove $D(\mathcal{A}) = \{\theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in \alpha_H S \ \forall H \in \mathcal{A}\}$

Geometric interpretation

$K = \mathbb{R}$ or $\mathbb{C}$

$\mathrm{Der}_S \longleftrightarrow$ polynomial vector fields on K^ℓ

$D(\mathcal{A}) \longleftrightarrow$ " " " on K^ℓ tangent to each hyperplane

$K = S_{(0)}$ the quotient field of S

Prop. $D(\mathcal{A}) \otimes_S K = \mathrm{Der}_S \otimes_S K \simeq K^\ell$

Pf: $Q\,\mathrm{Der}_S \subset D(\mathcal{A}) \subset \mathrm{Der}_S$

Def 1.4 $\mathcal{A}$ free $\Leftrightarrow D(\mathcal{A})$ is a free S-mod. (of rank ℓ)

$\theta_1, \dots, \theta_\ell \in D(\mathcal{A})$ homog. basis

$m_j = \deg \theta_j$

$$\mathrm{Der}_S = \bigoplus_{p=0}^{\infty} (\mathrm{Der}_S)_p$$

$$(\mathrm{Der}_S)_p = S_p d_1 \oplus \cdots \oplus S_p d_\ell$$

$m_1, \dots, m_\ell$ <u>exponents</u> of $\exp(\mathcal{A}) = (m_1, \dots, m_\ell)$

Prop. 1.5 $\theta_1, \dots, \theta_\ell \in D(\mathcal{A})$

$\Longrightarrow \det[\theta_i(x_j)] \in QS$

Pf: $H \in \mathcal{A}$ $\quad \alpha_H = \sum_{i=1}^{\ell} c_i x_i \quad (\exists j \text{ s.t. } c_j = 1)$

Then $\det[\theta_j(x_i)] =$

$$\det \begin{bmatrix} \theta_1(x_1) & \cdots & \theta_\ell(x_1) \\ \theta_1(\alpha_H) & \cdots & \theta_\ell(\alpha_H) \\ \theta_1(x_\ell) & \cdots & \theta_\ell(x_\ell) \end{bmatrix} \in QS$$

divisible by α_H

$\blacksquare$

Thm 1.6 (Saito's Criterion)

$\theta_1, \dots, \theta_\ell \in D(\mathcal{A})$ are a basis

$\Updownarrow$

$\det[\theta_j(x_i)] \doteq Q$

Cor. 1.7 $\exp(\mathcal{A}) = (m_1, \dots, m_\ell) \Rightarrow |\mathcal{A}| = m_1 + \cdots + m_\ell$

Thm 1.8 (Zaslavsky '75)

$\mathcal{A}$: any arr. $/ \mathbb{R}$

$$\mathrm{Poin}(\mathcal{A}, 1) = |\mathrm{Cham}(\mathcal{A})|$$

Theorem 1.9 (Factorization Thm '81)

$\exp(\mathcal{A}) = (m_1, \dots, m_\ell)$

$$\mathrm{Poin}(\mathcal{A}, t) = \prod_{i=1}^{\ell} (1 + m_i t)$$

Cor 1.10 If $\mathcal{A}$ is free then $|\mathrm{Cham}(\mathcal{A})| = \prod_{i=1}^{\ell}(1+m_i)$

Pr 2: Show that $Q = xyz(x+y+z)$ is not free a

Pr 3: Show that every 2-arr. is free.

Pr 4: $\Theta_E = \sum_{i=1}^{\ell} x_i \partial/\partial x_i \in D(\mathcal{A})$

Show that Θ_E can be part of a basis if $\mathcal{A}$ is non-empty. $\left(D(\mathcal{A}) = S\Theta_E \oplus \mathrm{Ann}(Q) \right)$

So, $\mathcal{A}$ non-empty free $\Rightarrow 1 \in \exp(\mathcal{A})$

2. Reflection Arr.

$\mathcal{A}$ irr. refl. arr. in V, $W = W(\mathcal{A})$

$R = S^W = \mathbb{R}[f_1, \dots, f_\ell]$ $f_1, \dots, f_\ell$ basic invariants

Def 2.1 $f \in S$, f is an antiinvariant

$\iff$ $w(f) = \det(w) f$
$w \in W$

Thm 2.2 $S^{-W} = \{\text{antiinvariant}\} = RQ$

Thm 2.3 $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (m_1, \dots, m_\ell)$
and irr. refl. arr.

$(m_j = \underline{\deg} f_j - 1)$ exponents of W

Pf: $f_1, \dots, f_\ell$ basic invariants

$$(\, , \,) : V^* \times V^* \longrightarrow \mathbb{R}$$

$$(\, , \,) : \Omega^1 \times \Omega^1 \to S$$

$$\theta_j(*) = (df_j, d*)$$

$\deg \theta_j = \deg f_j - 1$ if $df = \sum_{i=1}^{\ell} \frac{\partial f}{\partial x_i} dx_i$

θ_j is W-invariant, $\theta_j(Q) \in S^{-W} = RQ$

$\Rightarrow \theta_j \in D(\mathcal{A})$

$$\det[\theta_j(x_i)] = \det J(f_1, \dots, f_\ell) \doteq Q$$

by Saito we are done.

Pr 5: Find a basis for $D(B_\ell)$ $Q = x_1 \cdots x_\ell \prod (x_i^2 - x_j^2)$
Pr 6: $\quad$ " $\quad$ (D_ℓ) $\;$ " $= \prod (x_i^2 - x_j^2)$

Def 2.4: (unitary refl. grps) $V = \mathbb{C}^\ell$
$s \in U(\ell)$ is a unitary refl. $\iff \dim \mathrm{Fix}(s) = \ell - 1$

So, $s = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & \zeta \end{pmatrix}$ where $|\zeta| = 1$.

$G < U(\ell)$ a finite subgrp generated by unitary reflections irreducible

$S^G = \mathbb{C}[\underbrace{f_1, \dots, f_\ell}]$

homog. and alg. indep.

again

$m_j = \deg f_j - 1$ are the exponents of G.

$\mathcal{A}(G) = \{ \mathrm{Fix}(g) \mid g \in G \text{ is a unitary refl.} \}$

is unitary refl. arr.

Thm 2.5 $\mathcal{A}(G)$ is a free arr. with

$\exp(\mathcal{A}) = (n_1, \dots, n_\ell)$ are the coexp.

$\neq$

$\exp(G) = (m_1, \dots, m_\ell)$

Cor 2.6 $\mathrm{Poin}(\mathcal{A}(G), t) = \prod_{i=1}^{\ell} (1 + n_i t)$

3. What are free arrangements?

1. reflection arr.'s (real or complex) and their deformations
2. fiber type arr. (super-solvable)
3. arr.'s contracted by "addition - deletion thm" (inductively free arr.)

Addition-Deletion Thm

$H_0 \in \mathcal{A} \qquad \mathcal{A}' = \mathcal{A} \setminus \{H_0\}, \ \mathcal{A}'' = \mathcal{A}^{H_0} = \{H_0 \cap H \mid H \in \mathcal{A}'\}$

- $\mathcal{A}', \mathcal{A}''$ free w/ $\exp(\mathcal{A}'') \subset \exp(\mathcal{A}') \Rightarrow \mathcal{A}$ free and $\exp(\mathcal{A}'') \subset \exp(\mathcal{A})$ (addition)
- $\mathcal{A}, \mathcal{A}''$ free $\exp(\mathcal{A}'') \subset \exp(\mathcal{A}) \Rightarrow \mathcal{A}'$ free and $\exp(\mathcal{A}'') \subset \exp(\mathcal{A}')$ (deletion)

Pr 6 or 7 Show that $\mathcal{A}_\ell$ is inductively free with $\exp(\mathcal{A}_\ell) = (1, 2, \ldots, \ell)$

Conjecture (open) (1981) Does the K-fixed freeness of $\mathcal{A}$ only depend upon $L(\mathcal{A})$?

Falk

L supersolvable $\Longleftrightarrow \exists$ maximal chain of modular elements

semimodular : $rk(X \vee Y) + r(X \wedge Y) \le r(X) + r(Y)$

geometric lattice

$X \vee Y = X \cap Y \qquad X \wedge Y = \cap\{H \mid X + Y \subseteq H\}$

(X, Y) modular pair $\Longleftrightarrow$ equality holds

X is modular if (X, Y) is modular $\forall Y$

Thm: Suffices to check for only those Y for which $X \vee L = I_L$

Cor.: X modular $\Rightarrow (\forall Y, \ X \cap Y = 0 \Rightarrow X + Y = \mathbb{C}^\ell$

X modular $\Longleftrightarrow X + Y \in L \quad \forall Y$