Lecture 5 Terao

0 In the beginning

Def 0.1 V an ℓ-dim v.s. $/K$

$\mathcal{A}$ is a (central) arrangement of hyperplanes

$\Updownarrow$

$\mathcal{A}$ is a finite collection of one-codimensional vector subspaces of V.

Def 0.2 $L(\mathcal{A}) := \{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \}$ Agree $\bigcap_{H \in \emptyset} H = V$

poset $\quad X \leq Y \overset{\text{def.}}{\iff} X \supseteq Y$

V the minimum in $L(\mathcal{A})$

$T(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} H$ is the maximum of $L(\mathcal{A})$

Def 0.3 $\mu : L(\mathcal{A}) \to \mathbb{Z}$ defined inductively by

$$\mu(V) = 1$$

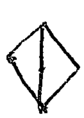
$$\mu(X) = - \sum_{\substack{Y \leq X \\ Y \neq X}} \mu(Y) \quad (X \neq V)$$

Pr 1: $\mathcal{A} = \mathcal{A}_4$ Find $T(\mathcal{A})$ and $\mu(T(\mathcal{A})) = 6$

Def 0.4

$$\text{Poin}(\mathcal{A}, t) = \sum_{X \in L} |\mu(X)| t^{\operatorname{codim} X}$$

$$= \sum_{X \in L} (-1)^{\operatorname{rk} X} \mu(X) t^{\operatorname{rk} X}$$

ex:  $\pi(\mathcal{A}, t) = 1 + 3t + 2t^2 = (1+t)(1+2t)$

1. Finite Reflection Grps

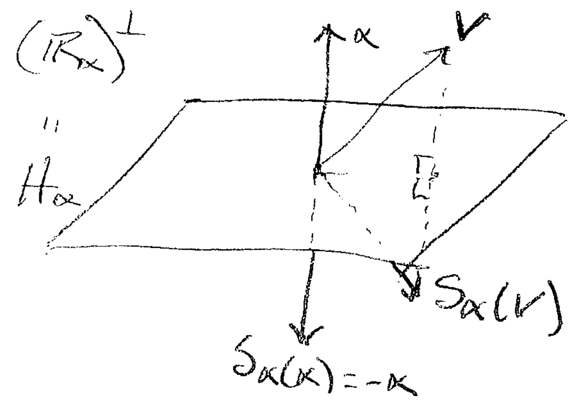
V ℓ-dim Euclidean space with inner product $(\,,\,)$ $/\mathbb{R}$

Let $\alpha \overset{\neq 0}{\in} V$

$S_\alpha \in O(V)$ $\begin{cases} S_\alpha(\beta) = \beta \text{ if } (\alpha,\beta)=0 \\ S_\alpha(\alpha) = -\alpha \end{cases}$

Then

$$\boxed{S_\alpha(v) = v - 2\frac{(\alpha,v)}{(\alpha,\alpha)}\alpha}$$



S_α: (orthogonal) reflection

$S_\alpha^2 = Id$

Notation

Def 1.1 $S_\alpha = S_{H_\alpha} = S_H$

$H = \ker(I - S_H) = Fix(S_H)$

Def 1.2 $W \subset O(V)$

W is finite reflection grp $\Leftrightarrow$ W is a finite subgrp of $O(V)$ generated by reflections

2. Reflection Arrangements

Def 2.1 $\mathcal{A}$ is an arr. of hyp. in V

$\mathcal{A}$ is reflection arr. $\Leftrightarrow$ $S_H(\mathcal{A}) = \mathcal{A}$ $\forall H \in \mathcal{A}$

$$\left\{\begin{array}{c}\text{finite}\\ \text{refl. grps}\end{array}\right\} \underset{\longleftarrow}{\xrightarrow{\text{one to one correspondence}}} \left\{\begin{array}{c}\text{refl.}\\ \text{arrang.}\end{array}\right\}$$

$$W \longmapsto \mathcal{A}(W) = \{\ker(1-s) \mid s \in W \text{ is a reflection}\}$$

$$W(\mathcal{A}) \longleftarrow\!\!\shortmid \mathcal{A}$$

$$W(\mathcal{A}) = \langle s_H \mid H \in \mathcal{A} \rangle$$

<u>Pr 2</u> Show $W(\mathcal{A})$ is a finite refl. grp if $\mathcal{A}$ is a refl. arr.

<u>Pr 3</u>: Which is the more obvious?

1. $\mathcal{A}(W(\mathcal{A})) = \mathcal{A}$ for any refl. arr.
2. $W(\mathcal{A}(W)) = W$ — — — — — — — grp

Prove the more obvious one and discuss the other.

<u>Examples</u> ($\mathcal{A}_\ell$) braid arr.

$e_1, \dots, e_{\ell+1}$ orthogonal basis for $\mathbb{R}^{\ell+1}$

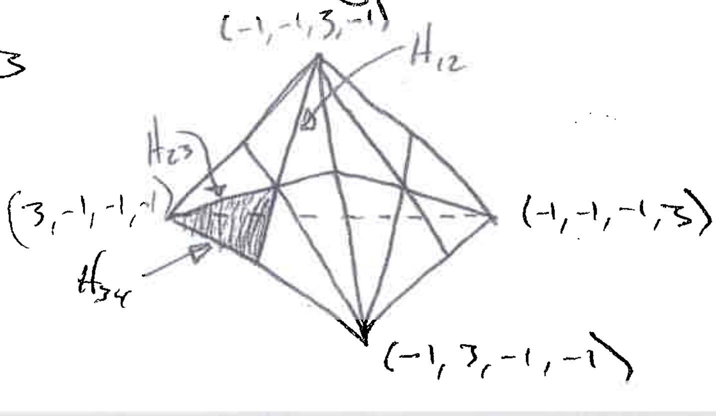
$x_1, \dots, x_{\ell+1}$ the dual " — — $(\mathbb{R}^{\ell+1})^*$

$$V = \ker(x_1 + \cdots + x_{\ell+1}) = \{c_1e_1 + \cdots + c_{\ell+1}e_{\ell+1} \in V \mid c_1 + \cdots + c_{\ell+1} = 0\}$$

$V \cong \mathbb{R}^\ell$

$$\mathcal{A}_\ell := \{H_{ij} \mid i < j\} \qquad H_{ij} = \ker(x_i - x_j) \cap V$$

$\mathcal{A}_3$



24 chambers

6 planes

<u>Pr 4</u>: Show $\mathcal{A}_\ell$ is a refl. arr. Find $|\mathcal{A}_\ell|$ and $|W(\mathcal{A})|$ and what grp is $W(\mathcal{A})$?

Example (B_ℓ) $\ell \geq 2$

$e_1, \dots, e_\ell$ O.N. basis of $\mathbb{R}^\ell = V$

$x_1, \dots, x_\ell$ dual basis

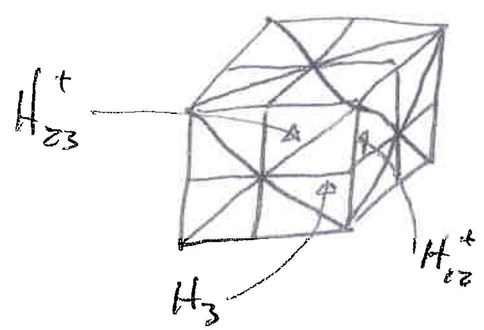
$H_{ij}^+ = \ker(x_i - x_j) \quad (i<j)$

$H_{ij}^- = \ker(x_i + x_j) \quad (i<j)$

$H_k = \ker(x_k)$

$$B_\ell = \{ H_{ij}^+, H_{ij}^-, H_k \mid 1 \leq i < j \leq \ell, 1 \leq k \leq \ell \}$$

Pr. 5 Same as Pr. 4 for B_ℓ



9 planes
48 chambers
$V = \mathbb{R}^3$

B_3

Example (D_ℓ) $\ell \geq 4$

Same notation as B_ℓ

$e_1, \dots, e_\ell$ $\quad H_{ij}^+, H_{ij}^-$

$$D_\ell = \{ H_{ij}^+, H_{ij}^- \mid 1 \leq i < j \leq \ell \}$$

Pr 6 Same as Pr 4 for D_ℓ

3. Chambers

$\mathcal{A}$ arr. in $\mathbb{R}^\ell = V$

__Def 3.1__ $M(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H$

A connected component of $M(\mathcal{A})$ is called a chamber.

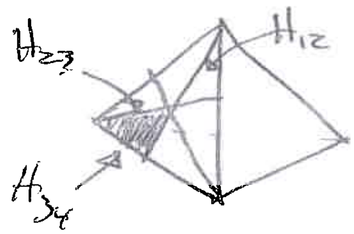
$\mathrm{Cham}(\mathcal{A}) = \{\text{chambers of } \mathcal{A}\}$

__Theorem 3.1__ Let $C \in \mathrm{Cham}(\mathcal{A})$ and $\mathcal{A}$ is a refl. arr. and $T(\mathcal{A}) = \{0\}$

$\Rightarrow \exists!\ H_1, \dots, H_\ell \in \mathcal{A}$
$\exists!\ \alpha_1, \dots, \alpha_\ell \in V$ s.t.

- $H_i = (\mathbb{R}\alpha_i)^\perp \quad \forall i$
- $\|\alpha_i\| = 1 \quad \forall i$
- $C = \{v \in V \mid (v, \alpha_i) > 0 \quad i = 1, \dots, \ell\}$

In particular, each chamber is a simplicial cone



A_3

$H_1 = H_{12} \quad \alpha_1 = \frac{1}{\sqrt{2}}(e_1 - e_2)$

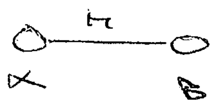
$H_2 = H_{23} \quad \alpha_2 = \frac{1}{\sqrt{2}}(e_2 - e_3)$

$H_3 = H_{34} \quad \alpha_3 = \frac{1}{\sqrt{2}}(e_3 - e_4)$

__system of symple roots__

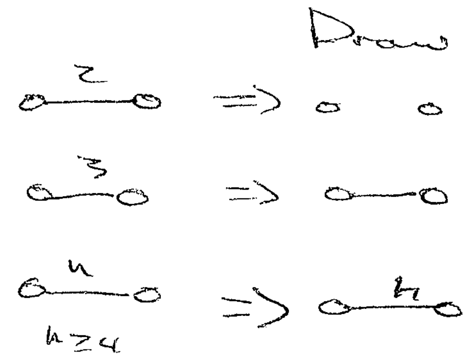
S.S.R.

Coxeter diagram

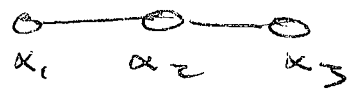


$\|\alpha\| = \|\beta\| = 1$

$(\alpha, \beta) = -\cos(\pi/4)$



For the example A_3



<u>Pr 7</u>: Find SSR and the Coxeter diagram for B_3

<u>Theorem 3.2</u>: $\mathcal{A}$ is refl. arr. and essential

$\alpha_1, \ldots, \alpha_\ell$ SSR

$\Rightarrow W = \langle s_{\alpha_1}, \ldots, s_{\alpha_\ell} \rangle$

<u>Thm 3.3</u>: W acts on $\mathrm{Cham}(\mathcal{A})$ transitively and effectively.

So, $\forall C_1, C_2 \in \mathrm{Cham}(\mathcal{A})$

$\exists!\, w \in W$ s.t. $C_1 = w(C_2)$

<u>Cor.</u> $|\mathrm{Cham}(\mathcal{A})| = |W(\mathcal{A})|$

4. Parabolic Subgrps

Recall $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \leq H\}$

<u>Thm 4.1</u>: $X \in L(\mathcal{A})$ $\quad W_X := \{w \in W \mid \mathrm{Fix}(w) \geq X\}$

$$\Rightarrow W_X = W(\mathcal{A}_X) \text{ and } \mathcal{A}(W_X) = \mathcal{A}_X$$

$$
\begin{array}{ccc}
\left\{\begin{array}{c}\text{finite}\\ \text{refl. grps}\end{array}\right\} & & \left\{\begin{array}{c}\text{refl.}\\ \text{arr.}\end{array}\right\} \\
W & \longmapsto & \mathcal{A}(W) \\
W(\mathcal{A}) & \longleftarrow\!\!\!\shortmid & \mathcal{A} \\
W_X & \longleftrightarrow & \mathcal{A}_X
\end{array}
$$

In particular, W_X is a finite refl. grp and $\mathcal{A}_X$ is a refl. arr.

<u>Thm 4.2</u> $L(\mathcal{A}) = \{\mathrm{Fix}(w) \mid w \in W\}$

<u>Pr. 8</u> $X \in L(\mathcal{A})$

$$|\mu(X)| = \left|\{w \in W \mid \mathrm{Fix}(w) = X\}\right|$$

$$\left(\text{Note: } \sum_{w \in W} \det w = 0\right)$$

5. Classification

Def 5.1 $\mathcal{A}_1$ arr. in V_1
$\mathcal{A}_2$ ---- V_2

$\Rightarrow$ $\mathcal{A}_1 \times \mathcal{A}_2 = \{H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2\}$

$|\mathcal{A}_1 \times \mathcal{A}_2| = |\mathcal{A}_1| + |\mathcal{A}_2|$

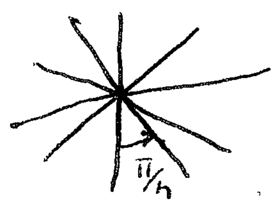
Def 5.2 $\mathcal{A}$ is called irreducible iff $\mathcal{A}$ can not be expressed as $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ with $\dim V_1 > 0$ & $\dim V_2 > 0$.

Thm 5.3: $\mathcal{A}$ is irr. refl. arr

$\Rightarrow$ $\mathcal{A}$ is one of the following

$\underline{l=1}$ $\mathcal{A}_1$

$\underline{l=2}$ $(n \geq 3)$ $I_2(n)$



$A_2 = I_2(3)$
$B_2 = I_2(4)$
$G_2 = I_2(6)$

$\underline{l=3}$ $\mathcal{A}_3$, B_3, H_3

$\underline{l=4}$ $\mathcal{A}_4$, B_4, D_4, F_4, H_4

$\underline{l \geq 5}$ A_l, B_l, D_l
E_6
E_7
E_8

Pr 9: $V = \langle e_1, \dots, e_\ell \rangle$

$$[(e_i, e_j)] = \frac{1}{2} \underbrace{\begin{pmatrix} \frac{8}{9} & -\frac{1}{9} & \cdots & -\frac{1}{9} \\ -\frac{1}{9} & \frac{8}{9} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ -\frac{1}{9} & \cdots & & \frac{8}{9} \end{pmatrix}}_{\ell}$$

$$\mathcal{A}_\ell = \{\ker(x_i - x_j)\} \cup \{\ker(x_i + x_j + x_k)\}$$
$$\cup \{\ker(x_{i_1} + \dots + x_{i_6}) \mid 1 \le i_1 < \dots < i_6 \le \ell\}$$

$E_6 = \mathcal{A}_6$, $E_7 = \mathcal{A}_7$

$\underset{SSR}{\Delta} = \{e_i - e_{i+1} \mid i = 1, \dots, \ell-1\} \cup \{e_{\ell-2} + e_{\ell-1} + e_\ell\}$

Find $|E_6|$ & $|E_7|$. Show E_6 & E_7 are refl. arr. Find the Coxeter diagrams of E_6 & E_7.

6. Basic Invariants

$\mathcal{A}$ refl. arr. in V, $W = W(\mathcal{A})$, V^*-dual

Def 6.1 $W \curvearrowright V^*$ contragradient action

$\langle wx, v \rangle = \langle x, w^{-1} v \rangle \quad x \in V^*, v \in V, w \in W$

Def 6.2 $S := S(V^*)$ the symmetric alg. of $V^* / \mathbb{R}$

Then $S = \mathbb{R}[x_1, \dots, x_\ell]$ is polynomials on V

$S = \bigoplus_{p=0}^{\infty} S_p \qquad S_1 = V^* \qquad W \curvearrowright S_p$

Thm 6.3 (Shephard-Todd 54, Chevalley 55)

$\mathcal{A}$ is (irr.) refl. arr.

$R = S^W$ the invariant subring

$\Rightarrow$ (1) $\exists\ \underbrace{f_1, \dots, f_\ell}_{\text{homog. alg. indep.}} \in R$ s.t. $R = \mathbb{R}[f_1, \dots, f_\ell]$ — {basic invariants

(2) $\exists\ U \subset S$ s.t. $S = U \otimes R$
W-stable

(3) $W \curvearrowright U$
regular representation

$$\begin{array}{ccc} \mathbb{C}[x_1, \dots, x_\ell] & & \\ \cup & \xrightarrow{\text{Spec}} & \\ \mathbb{C}[x_1, \dots, x_\ell]^W = \mathbb{C}[f_1, \dots, f_\ell] & & \end{array}$$

$$\begin{tikzcd} & \mathbb{C}^\ell \arrow[dl] \arrow[dr, "F=(f_1,\dots,f_\ell)"] & \\ \mathbb{C}^\ell/W \arrow[rr, phantom, "\cong"] & & \mathbb{C}^\ell \end{tikzcd}$$

the orbit space

$$\operatorname{crit}(F) = \bigcup_{H \in \mathcal{A}} H$$

the set of critical pts

$$\| $$

$$\{ J(f_1, \dots, f_\ell) = 0 \}$$

$$J(f_1, \dots, f_\ell) = \det\left[\frac{\partial f_i}{\partial x_j}\right]$$

Vandermonde determinate

$$\mathbb{C}^\ell = \{ (x_1, \dots, x_{\ell+1}) \mid x_1 + \dots + x_{\ell+1} = 0 \}$$

$$\Big\downarrow (f_1, \dots, f_\ell) \qquad f_j = \frac{1}{j+1} \sum_{k=1}^{\ell+1} x_k^{j+1} \quad \text{power sum}$$

$$\mathbb{C}^\ell$$

$$\begin{vmatrix} x_1 & x_1^2 & \cdots & x_1^\ell \\ x_2 & & & \\ \vdots & & & \\ x_\ell & x_\ell^2 & \cdots & x_\ell^\ell \end{vmatrix} = \prod (x_i - x_j)$$

<u>Exponents</u> $f_1, \dots, f_\ell$ basic invariants

$m_j = \deg f_j - 1 \quad (j = 1, \dots, \ell)$

$\mathcal{R}$ <u>irred.</u> refl. arr.

$$1 = m_1 < m_2 \le \cdots \le m_{\ell-1} < m_\ell$$

$h = m_\ell + 1$ is the <u>Coxeter #</u>

- $m_j + m_{\ell+1-j} = h \quad (j = 1, \dots, \ell)$

$$\mathrm{Poin}(\mathcal{R}, t) = (1 + t^{m_1}) \cdots (1 + t^{m_\ell})$$