

\mathcal{L} fiber type

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Cor. $\bigcap_{n \geq 1} G_n = 1$

i.e. G is residually nilpotent.

Suciu

LCS: $G = G_1 > G_2 > \dots > G_r > \dots$

$\text{rank } G_r = \mathcal{O}_R$

LCS Quotients G_r / G_{r+1}

Nilpotent quotients G / G_r

$G / G_2 = G^{ab} = \mathbb{Z}^m$

$G_2 / G_3 \rightarrow G / G_3 \rightarrow G / G_2 \rightarrow 1$

$G_r / G_{r+1} \rightarrow G / G_{r+1} \rightarrow G / G_r \rightarrow 1$

ex: $G = F_2 = \langle x, y \rangle$

$G_2 / G_3 \rightarrow G / G_3 \xrightarrow{ab} G / G_2 \rightarrow 1$
 \parallel \parallel
 $[x, y]$ \mathbb{Z}^2
 x, y

$$G/G_3 = \langle x, y \mid [x, [x, y]] = [y, [x, y]] = 1 \rangle \quad (111)$$

$$\text{Heisenberg } \mathfrak{g}_3^* = \left\{ \begin{pmatrix} 1 & x & c \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$$

Thm: $G = \pi_1(M)$
 $K = \pi_1(N)$

$$G/G_3 \cong K/K_3 \iff H^{\leq 2}(M) \cong H^{\leq 2}(N)$$

(in particular G/G_3 is combinatorially determined)

Warning G/G_4 is not comb. determined.

Let
$$\nu_{p,d}(G) := \# \left\{ K \triangleleft G/G_3 \mid \begin{array}{l} [G/G_3 : K] = p \\ \dim_{\mathbb{Z}_p} H^1(K, \mathbb{Z}_p) = u+d \end{array} \right\}$$

p is prime, $d \geq 0$

Thm:
$$\nu_{p,d}(G) = \frac{1}{p-1} \left| \mathbb{F}_d(G, \mathbb{Z}_p) \right| \left| \mathbb{F}_{d+1}(G, \mathbb{Z}_p) \right|$$

Associated graded Lie alge

$$\mathfrak{g}^r(G) = \bigoplus_{r=1}^{\infty} G_r$$

$\underbrace{G_{r+1}}_{\mathfrak{g}^{r+1}(G)}$

$$[,] : \mathfrak{g}^r_k(G) \times \mathfrak{g}^r_s(G) \rightarrow \mathfrak{g}^r_{k+s}(G)$$

given by commutator $[x, y] = xyx^{-1}y^{-1}$

$[x, y] = -[y, x]$ and Jacobi identity

Ex: $G = F_n$ free grp

$\mathfrak{g}^r(G) = L_n$ free Lie alge.

$U(L_n) = T_n =$ tensor alge.

\uparrow
univ. enveloping alge. $\phi_n = \text{rank } \mathfrak{g}^r(G)$

P.B.W. Thm

$$\prod_{r=1}^{\infty} (1 - z^r)^{-\phi_r} = \text{Hilb}(U(\mathfrak{g}^r(G)))$$

Ex: $G = F_n$ $\prod_{r=1}^{\infty} (1 - z^r)^{-\phi_r(F_n)} = \text{Hilb}(T_n) = \frac{1}{1-nz}$

Chen Lie alg (~1950)

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$$\underline{gr}(G/G'') \quad \Theta_r = \text{rank } \underline{gr}_r(G/G'')$$

this is a quotient of $gr(G)$, so

$$\Phi_r \geq \Theta_r \quad \text{also} \quad \Phi_i = \Theta_i \quad \text{for } i=1,2,3$$

Theorem: (Massey)
 $\Theta_r = \dim_{\mathbb{C}} \underline{gr}_{r-2} B \otimes \mathbb{C}$
($B = G'/G''$ Alex inv.)

Filtered by I^r

$$\underline{gr}_r B = \frac{I^r B}{I^{r+1} B}$$

in particular

$$\Phi_2 = \Theta_2 = \dim \frac{B}{I B}$$

$$\Phi_3 = \Theta_3 = \dim \frac{I B}{I^2 B}$$

Theorem (PSO)

$$\Theta_r = \dim \underline{gr}_{r-2} B^{\text{lin}} \otimes \mathbb{C}$$

combinatorially determined

Application

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$$\text{Chem} \cdot \Theta_r(F_n) = (r-1) \binom{n+r-2}{r} \quad r \geq 2$$

$$\text{C.S. 94} \quad \Theta_r(P_n) = (r-1) \binom{n+1}{4} \quad \text{for } r \geq 2$$

$$\# \quad \Theta_r(F_{n_1} \times \dots \times F_i) \quad \text{for } n \geq 4$$

Digression

Recall

$$R_d(G_n) = V(\text{ann } B^{\text{dim}})$$

Question: Can we write Θ_r (and maybe Φ_r) in terms of $R_d(\sigma \mathbb{R})$?

Resonance L.G.S. Conjectures (2000)

$$R_1 = R_1^{\text{linear}} = \bigcup_{i=1}^s L_i, \quad L_i \text{ subspace of } \mathbb{C}^n$$

$$h_r := \# \{L_i \mid \dim L_i = r\} \quad \begin{array}{l} \text{[for non-resonant arrs]} \\ \text{[for graph arrs]} \end{array}$$

Then:

[true (SS 04)]

$$\bullet \quad \Theta_k = \sum_{r \geq 2} h_r \cdot \Theta_k(F_r) \quad \text{for } k \gg 0.$$

$$\bullet \quad \Phi_k = \sum_{r \geq 2} h_r \Phi_k(F_r) \quad \text{if } \Theta_4 = \Phi_4 \text{ for } k \gg 0$$

Thm: If $\phi_3 = \sum_{x \in L_2(\Omega)} \phi_3(F_{\mu(x)})$ then

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$$\prod_{r=1}^{\infty} (1 - z^r)^{\phi_r(\Omega)} = (1 - z)^{|\Omega| - \sum_{x \in L_2(\Omega)} \mu(x)} \prod_{x \in L_2(\Omega)} (1 - \mu(x)z)$$

THE END