Applications of Characteristic Varieties

1. $1 \to K_\gamma \to G \xrightarrow{\gamma} \mathbb{Z}_N \to 1$

$\longleftrightarrow$ $\mathbb{Z}_N \to X^\gamma \to X$ cyclic regular N-fold cover of $X = X_G$

$\mathbb{K}$-field (sufficiently large w.r.t. $\mathbb{Z}_N$)

Then
$$\boxed{\dim_{\mathbb{K}} H^1(X^\gamma, \mathbb{K}) = n + \sum_{1 \neq k \mid N} \phi(k) \cdot \operatorname{depth}_{\mathbb{K}}(\gamma^{N/k})}$$

where
$$\operatorname{depth}_{\mathbb{K}}(t) = \max\{d \mid t \in V_d(G, \mathbb{K})\}$$
for $t \in \operatorname{Hom}(G, \mathbb{K}) \cong (\mathbb{K}^*)^n$

$$\begin{tikzcd} G \arrow[r, "\gamma"] \arrow[rrr, bend right, "\gamma^r"'] & \mathbb{Z}_N \arrow[r, hook] & \mathbb{K}^* \arrow[r, "x \mapsto x^r"] & \mathbb{K}^* \end{tikzcd}$$

$1 \longmapsto$ primitive N^{th} root of 1

2. Counting homomorphisms to finite grps

$$\delta_\Gamma(G) = \frac{|\mathrm{Epi}(G, \Gamma)|}{|\mathrm{Aut}(\Gamma)|}$$
G-finitely gen
Γ-finite

$(= GQuotients(G,\Gamma)$ in $Gap)$

Use characteristic varieties of G to compute these #'s, in case Γ is metabelian, e.g. $\Gamma = S_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$

$\Gamma = A_4 = (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_3$

$$\delta_{S_3}(G) = \frac{1}{2} \sum_{d \geq 1} \left| \mathrm{Tors}_2\left(V_d(G,\mathbb{Z}_3) \setminus V_{d+1}(G,\mathbb{Z}_3)\right)\right| \cdot (3^d - 1)$$

$$\delta_{A_4}(G) = \frac{1}{6} \sum_{d \geq 1} \left| \mathrm{Tors}_3\left(V_d(G,\mathbb{F}_4) \setminus V_{d+1}(G,\mathbb{F}_4)\right)\right| \cdot (4^d - 1)$$

where $\mathrm{Tors}_p(V) = \{t \in V \mid t^p = 1,\ t \neq 1\}$ for $V \subset (\mathbb{k}^*)^n$

Milnor fibration

$\mathcal{A} = \{H_1, \dots, H_n\}$ arr. in $\mathbb{C}^\ell$, $H_i = \ker \alpha_i$ $\quad \alpha_i : \mathbb{C}^\ell \to \mathbb{C}$ linear

$M = \mathbb{C}^\ell \setminus \cup H_i \quad \overline{M} = \mathbb{CP}^{\ell-1} \setminus \cup \overline{H}_i$

Fix weights $a = (a_1, \dots, a_n)$ $a_i \geq 1$ $\in \mathbb{Z}_{>0}$ set $N = \sum_{i=1}^{n} a_i$

$f : \mathbb{C}^\ell \longrightarrow \mathbb{C}$

$f = \alpha_1^{a_1} \cdots \alpha_n^{a_n}$ (homog. poly. of deg. N)

The restriction of f, $f : M \longrightarrow \mathbb{C}^*$ is the projection of a fiber bundle

(i.e. a submersion)

with • fiber $F = f^{-1}(1)$

• monodromy $h: F \to F$

$h(z_1, \dots, z_\ell) = (\zeta z_1, \dots, \zeta z_\ell)$

$\zeta = e^{2\pi i/N}$

(variant: $f/|f| : M \to S^1$)

Basic Questions: Note: $h^N = Id$

• What is $H_*(F)$

• what is $h_* : H_*(F) \to H_*(F)$

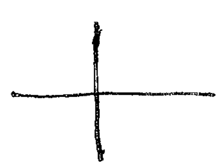
In Particular Question: What is $H_*(F)$ when $\mathcal{A}$ = braid arr. & weights $= 1$

$F = \{ \prod_{i<j} (z_i - z_j) = 1 \}$

Quick example

(i) $f: \mathbb{C}^2 \to \mathbb{C}$ $\quad f(x,y) = xy$

$\mathcal{A}$

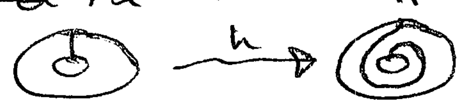


$$\begin{array}{ccc} F = \mathbb{C}^* & \to & M \\ & & \downarrow f \\ & & \mathbb{C}^* \end{array}$$

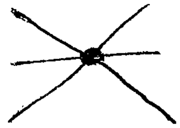
$H_1(F) = \mathbb{Z}$

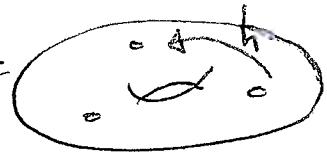
$h_* : \mathbb{Z} \to \mathbb{Z}$ $\quad h_* = Id$

But $h \neq id$ and in fact h = Dehn twist



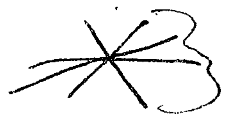
(ii) $f(x,y) = xy(x-y)$

$\mathcal{A}$ 

$F =$ 

Problem: compute $H_*(F)$ (Hint: $H_1(F) = \mathbb{Z}^4$)
and $h_* : H_*(F) \to H_*(F)$

More generally

$\mathcal{A}$  n $\quad H_1(F) = \mathbb{Z}^{(n-1)^2}$

Recall Hopf fibration

$$\begin{tikzcd}
\mathbb{Z}_N & \mathbb{C}^* \arrow[d] & & \mathbb{C}^* \arrow[d] \\
F \subset \arrow[dr, "\pi_F"'] & M \arrow[d, "\pi"] & \subset & \mathbb{C}^\ell \setminus \{0\} \arrow[d, "\pi"] \\
& \bar{M} \arrow[u, bend right, "s"'] & \subset & \mathbb{CP}^{\ell-1}
\end{tikzcd}$$

Putting things together

$$\begin{tikzcd}
& & \mathbb{Z}_N \arrow[d] & \\
\mathbb{Z}_N \arrow[dr] & & F \arrow[d] \arrow[dr, "\pi_F"] & \\
& \mathbb{C}^* \arrow[r] \arrow[dr, "z \mapsto z^N"'] & M \arrow[r] \arrow[d] & \bar{M} \arrow[l, bend left, "s"] \arrow[dl, "\varphi"] \\
& & \mathbb{C}^* &
\end{tikzcd}$$

ie. $\begin{tikzcd} \mathbb{Z}_N \arrow[r] & F \arrow[d] \\ & \bar{M} \end{tikzcd}$ is pullback of cannonical N-fold cover $\mathbb{Z}_N \to \mathbb{C}^* \to \mathbb{C}^*$

Passing to homotopy

$$\begin{tikzcd}
& & & 1 \arrow[d] & 1 \arrow[dl] & \\
& & & \pi_1(F) \arrow[dl] \arrow[d] & & 1 \arrow[dl] \\
& 1 \arrow[r] & \pi_1(\bar{M}) \arrow[dl] \arrow[r, shift right, "s_*"'] & \pi_1(M) \arrow[l, shift right, "\pi_*"'] \arrow[d] \arrow[r, shift left] & \mathbb{Z} \arrow[l, shift left, bend right] \arrow[r] \arrow[dl, "\cdot N"] & 1 \\
& \mathbb{Z}_N \arrow[dl] & & \mathbb{Z} \arrow[d] \arrow[dl] & & \\
1 & & \mathbb{Z}_N \arrow[dl] & 1 & & \\
& 1 & & & &
\end{tikzcd}$$

Covering $\begin{array}{c} F \\ \downarrow \\ \bar{M} \end{array}$ is classified by

$$\gamma : \pi_1(\bar{M}) \to \mathbb{Z}_N$$
$$x_i \longmapsto a_i$$

From result before, we get

$$\boxed{\dim_{\mathbb{K}} H_1(F, \mathbb{K}) = n - 1 + \sum_{1 \neq k | N} \phi(k) \cdot \operatorname{depth}_{\mathbb{K}} \left(\gamma_a^{N}\right.}$$

Example 1

$\mathcal{A} =$ $f(x,y) = xy(x-y)$

$t_1 = w$, $t_2 = w$, $(t_1t_2)^{-1} = (w^2)^{-1}$

$V_1 = \{t_1t_2t_3 - 1 = 0\} = \{(t_1, t_2, (t_1t_2)^{-1})\}$

$V_2 = \{1\}$

$N=3$

$$b_1(F) = 2 + \underset{\substack{\shortparallel \\ 2}}{\phi(3)} \cdot \underset{\substack{\shortparallel \\ 1}}{\mathrm{depth}(\gamma^1)} \qquad \gamma = \gamma_{(1,1,1)} = (\omega,\omega,\omega) \quad \omega = e^{2\pi i/3}$$

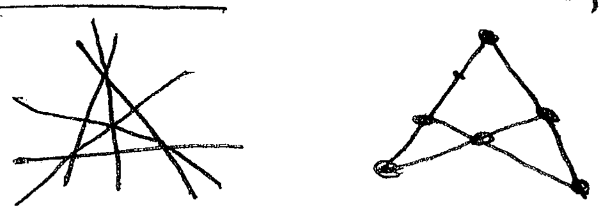
$$= 4$$

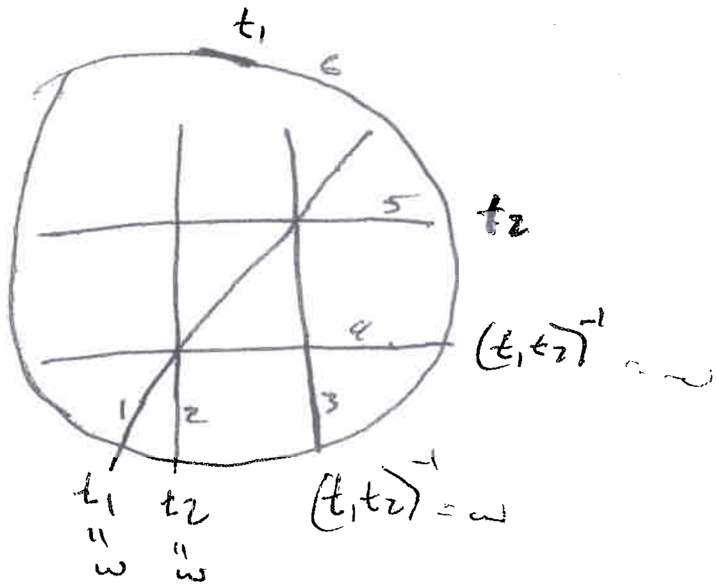
$$\delta_{S_3}(G) = \frac{1}{2}\Big(\underset{2^2-1}{\mathrm{Tors}_2 V_1(G,\mathbb{Z}_3)\setminus V_2(G,\mathbb{Z}_3)}\Big)(3-1) = 3$$

$$\delta_{A_4}(G) = \frac{1}{6}\Big(\underset{3^2-1}{\mathrm{Tors}_3\big(V_1(G,\mathbb{Z}_3)\setminus V_2(G,\mathbb{Z}_3)\big)}\Big)(4-1) = 4$$

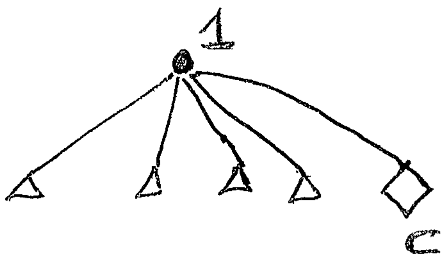
<u>Example 2</u> (Braid arr.)

$f = x_1 x_2 x_3 (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$





Char. Varieties



$$C = \{t_1 = t_6,\ t_3 = t_4,\ t_2 = t_5,\ t_1 \cdots t_6 = 1\}$$

$$F = f^{-1}(1) \qquad \gamma = \gamma_a : \pi_1(\bar{M}) \to \mathbb{Z}_6 \subset k^*$$

$$\gamma = (\zeta, \dots, \zeta) \qquad \zeta = e^{2\pi i/6}$$

$$b_1(F) = 5 + \underset{\substack{\shortparallel\\ 1}}{\phi(2)}\,\underset{\substack{\shortparallel\\ 0}}{\cancel{\mathrm{depth}}(\gamma^3)} + \underset{\substack{\shortparallel\\ 2}}{\phi(3)}\,\underset{\substack{\shortparallel\\ 1}}{\cancel{\mathrm{depth}}(\gamma^2)}$$

$$+ \underset{\substack{\shortparallel\\ 2}}{\phi(6)}\,\underset{\substack{\shortparallel\\ 0}}{\cancel{\mathrm{depth}}(\gamma)}$$

$$\zeta = \omega$$

$$= 7$$

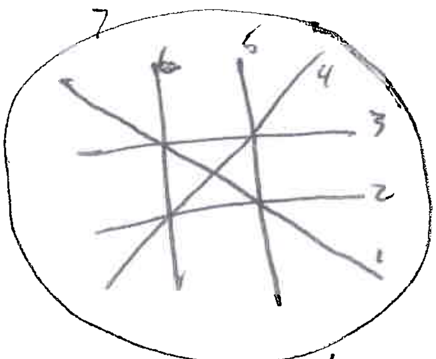
$$\delta_{S_3} = 5\cdot 3 = 15$$

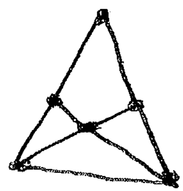
in fact $H_1(F) = \mathbb{Z}^7$

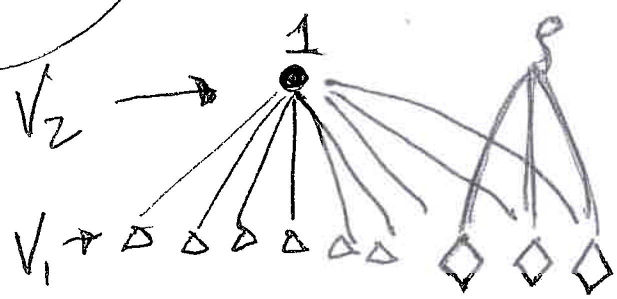
$$\delta_{A_4} = 5\cdot 4 = 20$$

<u>Ex: 3</u> (non-fano)

$$f = xyz(x-y)(x-z)(y-z)(x+y-z)$$







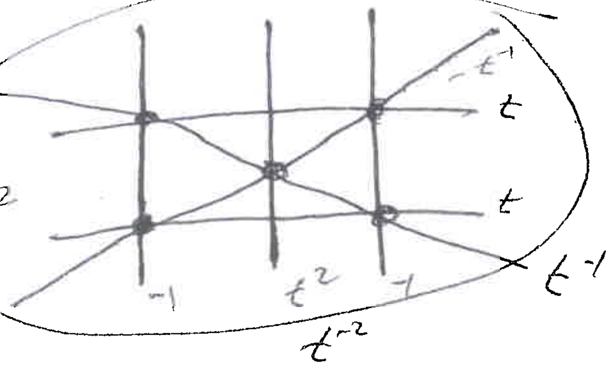
$$\rho = (1, -1, -1, 1)$$

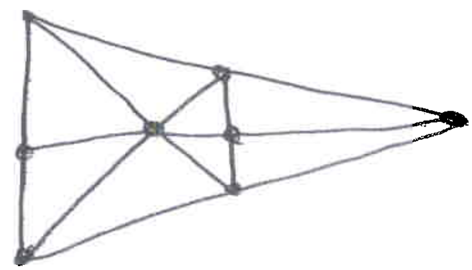
$$b_1(F) = \underbrace{6}_{\#\text{ of components}} + \underset{\substack{\shortparallel\\ 6}}{\phi(7)}\,\underset{\substack{\shortparallel\\ 0}}{\mathrm{depth}(\gamma)} = 6$$

$$\delta_{S_3}(G) = \underset{\substack{\shortparallel\\ 27}}{\tfrac{1}{2}\,9\cdot(3-1)(2^2-1)}$$ [redacted] $$-3 + \tfrac{1}{2}(3^2-1) =$$

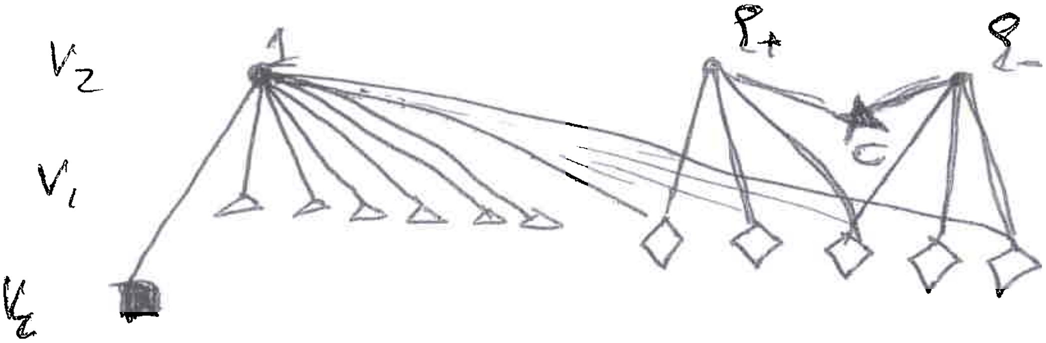
$$\delta_{A_4}(G) = 9\cdot 4 = 36$$

Example 4 (Deleted B_3)





$$Q = x_1 x_2 (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2)$$



C = 1-dim. torus which does not go thru 1

$C = \{(t^2, t^{-2}, -1, -1, t^{-1}, -t^{-1}, t, t)\} = \rho \cdot T$ $\quad \rho^2 = 1 \quad \rho \neq 1$

<u>Milnor fibration with</u> $a = (2,1,3,3,2,2,1,1)$

$N = 15$ decone w.r.t. x_2

$F \xrightarrow{\mathbb{Z}_{15}} M$ given by $\gamma_a = (\zeta^2, \zeta, \zeta^3, \zeta^3, \zeta^2, \zeta^2, \zeta, \zeta)$

where $\zeta = e^{2\pi i/15}$

$b_1(F, \mathbb{K}) = 7 + \phi(3) d(\gamma^5) + \phi(5) d(\gamma^3) + \phi(15) d(\gamma)$

if char $\mathbb{K} \neq 2$ $\quad b_1(F, \mathbb{K}) = 0 + 7 \quad \gamma^5 = (\omega^2, 1, 1, \omega^2, \omega^2, \omega, \omega) \in C$

'' '' '' $= 2$ $\quad b_1(F, \mathbb{K}) = 9$

in fact $\quad H_1(F) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

(49)

$$\delta_{S_3}(G) = \frac{1}{2}(2^3-1)(3^2-1) + 11 \cdot \frac{1}{2}(2^2-1)(3-1)$$

$$-6 + 2 \cdot \frac{1}{2}(3^2-1) = 63$$

$$\delta_{A_4}(G) = \frac{1}{6}(3^3-1)(4^3-1) + 11 \cdot \frac{1}{6}(3^2-1)(4-1)$$

$$+ 1 \cdot \frac{1}{6}(3-1)(4-1)$$

$$= 110$$