Lecture 13 Suciu (75)

Characteristic Varieties of Arr's

§0 Intro

$\mathcal{A}$ arr. $|\mathcal{A}|=n$, $M(\mathcal{A})$: complement

$G=\pi_1(M(\mathcal{A}))$ $\mathbb{K}$-field (alg. closed $\mathbb{K}=\mathbb{C}$)

Char. Varieties of $\mathcal{A}$ over $\mathbb{K}$

$$V_d^i(\mathcal{A},\mathbb{K}) := \{t\in(\mathbb{K}^*)^n \mid \dim_{\mathbb{K}} H^i(M, \mathbb{K}_t)\geq d\}$$

rank 1 local system defined by t

$$(\mathbb{K}^*)^n = V_0^i \supset V_1^i \supset V_2^i \supset \cdots \supset V_n^i$$

Resonance Varieties

$$R_d^i(\mathcal{A},\mathbb{K}) = \{a\in\mathbb{K}^n \mid \dim_{\mathbb{K}} H^i(H^*(M,\mathbb{K}), \cdot a)\geq d\}$$

Object of the talk:

- interpret $V_d(\mathcal{A}) := V_d^1(\mathcal{A},\mathbb{K})$ and $R_d(\mathcal{A}) := R_d^1(\mathcal{A})$ in concrete te
- relate these two varieties
- Applications
  - homology of finite cover
  - Milnor fibration.

- homology of Milnor fiber
- # of epi's to finite grps
- counting finite index sbgps

1. Homology of finite cover

$\overset{\Phi}{\twoheadrightarrow} G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle \; , \; r_i \in [F_n, F_n]$

$\langle x_1, \dots, x_n \rangle$

X = 2-complex modelled in this presentation

$\tilde{X}$ = universal cover

$$\begin{array}{rcccccccl}
\tilde{C}_*(\tilde{X}): & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\varepsilon} & \mathbb{Z} & \to 0 \\
& \| & & \| & & \| & & & \\
& (\mathbb{Z}G)^m & \xrightarrow[J_G^{ab} = \left(\frac{\partial r_i}{\partial x_j}\right)]{} & (\mathbb{Z}G)^n & \xrightarrow[\begin{pmatrix} x_1 - 1 \\ \vdots \\ x_n - 1 \end{pmatrix}]{} & \mathbb{Z}G & \to & \mathbb{Z} & \to 0
\end{array}$$

Sps $1 \to K_\gamma \hookrightarrow G \overset{\gamma}{\twoheadrightarrow} \Gamma \to 1 \quad \Longleftrightarrow \quad \Gamma \to X^\gamma$
$\downarrow$
X

reg. Γ-cover with $\pi_1(X^\gamma) = K_\gamma$

We have:

$$\begin{array}{rcccccl}
C_*(\tilde{X}^\gamma): & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 & \\
& \| & & \| & & \| & \\
& (\mathbb{Z}\Gamma)^m & \xrightarrow[J_G^\gamma]{} & (\mathbb{Z}\Gamma)^n & \xrightarrow[\begin{pmatrix} \gamma(x_1) - 1 \\ \vdots \\ \gamma(x_n) - 1 \end{pmatrix}]{} & \mathbb{Z}\Gamma & \quad (*)
\end{array}$$

(77)

ex: $1 \to G' \to G \xrightarrow{ab} \mathbb{Z}^n \to 1$ $\quad X^\gamma = X^{ab}$ ab. cov

Λ = Laurent poly $= \mathbb{Z}\mathbb{Z}^n$

$C_*(X^{ab})$: $\Lambda^m \longrightarrow \Lambda^n \longrightarrow \Lambda$

$M = J_G^{ab}$ $\quad \begin{pmatrix} t_1 - 1 \\ \vdots \\ t_n - 1 \end{pmatrix}$

M = alex. matrix

ex: Γ finite, $|\Gamma| = k$,

(*) becomes

$$\mathbb{Z}^{km} \xrightarrow[J^\gamma]{} \mathbb{Z}^{kn} \xrightarrow[\partial_1^\gamma]{} \mathbb{Z}^k$$

where $G \longrightarrow \mathrm{Sym}\left(\overset{\Gamma}{\underset{\shortparallel}{}} G/K_\gamma\right) \cong S_k \hookrightarrow GL(k, \mathbb{Z})$

coset repr. $\qquad$ perm. rep.

Problem 1: $H_1(K_\gamma) \oplus \mathbb{Z}^{k-1} \cong \operatorname{coker}(J^\gamma)$

Now, sps $\mathbb{K}$ is sufficiently large w.r.t. Γ (either char $\mathbb{K} = 0$ or char $\mathbb{K} \nmid |\Gamma|$

Then: $$\dim_{\mathbb{K}} H_1(K_\gamma, \mathbb{K}) = n + \sum_{\substack{\rho \in \text{irr. rep} \\ \rho \neq 1}} n_\rho (\operatorname{corank} J^{\rho\circ\gamma} - n_\rho)$$

where $n_\rho = \dim \rho$

$\rho : \Gamma \longrightarrow GL(n_\rho, \mathbb{K})$

2. Characteristic Varieties of G

$$\hat{G} = \mathrm{Hom}(G, \mathbb{K}^*) \cong \mathrm{Hom}(\mathbb{Z}^n, \mathbb{K}^*) \cong (\mathbb{K}^*)^n$$

character torus

$$G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$$

$(\mathbb{K}^*)^n \ni t = (t_1, \dots, t_n) \longleftrightarrow \mathbb{K}_t^* =$ rank 1 ~~module~~ local system

i.e. $\mathbb{K}$ viewed as a $\mathbb{Z}G$-mod.

$$G \xrightarrow{ab} \mathbb{Z}^n \longrightarrow \mathbb{K}^* = GL(1, \mathbb{K})$$

i^{th} basis element $\longmapsto t_i$

$$C_*(X, \mathbb{K}_t) = C_*(\hat{X}) \otimes_{\mathbb{Z}G} \mathbb{K}_t :$$

which gives

$$\mathbb{K}^m \xrightarrow[\parallel]{\partial_2^t} \mathbb{K}^n \xrightarrow[\parallel]{\partial_1^t} \mathbb{K}$$

$M(t)$ $\quad$ $\begin{pmatrix} t_1 - 1 \\ \vdots \\ t_n - 1 \end{pmatrix}$

Alex. matrix evaluated at t

Then $$H_1(G, \mathbb{K}_t) = H_1(X, \mathbb{K}_t) = \ker(\partial_1^t) / \mathrm{Im}(\partial_2^t)$$

Remark $$H^1(G; \mathbb{K}_t) = H_1(G, \mathbb{K}_{t^{-1}})$$

Def: $V_d(G,\mathbb{k}) = \{ t \in (\mathbb{k}^*)^n \mid \dim_{\mathbb{k}} H^1(G, \mathbb{k}_t) \geq d \}$

From the above discussion we get:

$$\boxed{V_d = \{ t \mid \operatorname{rank} M(t) < n-d \}}$$

= zero locus of ideal of codim d minor of Alex. matrix specialized at t.

— a subvariety of char. torus

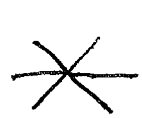
Problem 2: If $G = \pi_1(M(\mathcal{A}))$, then

$$V_d(G) = V_d(\mathcal{A}) \text{ and } R_d(G) = R_d(\mathcal{A})$$

[Hint: Use the fact that $\underset{\text{Hurewicz}}{h}: \pi_2 M \to H_2 M$ is zero + Hopf exact seq. $\pi_2 X \to H_2 X \to H_2 G \to 0$

Problem 3: Show that all entries in the Alex. matrix of an arr. lie in the augmentation ideal of $\Lambda = \mathbb{Z}\mathbb{Z}'$

Ex

$\mathcal{A} =$ 

$$G = \left(x_1, x_2, x_3 \;\middle|\; \begin{matrix} x_1x_2x_3 = x_3x_1x_2 \\ = x_2x_3x_1 \end{matrix} \right) \cong F_1 \times F_2$$

$$M = J_G^{ab} = \begin{bmatrix} 1-t_3 & t_1 - t_3t_2 & t_1t_2 - 1 \\ 1 - t_2t_3 & t_1 - 1 & t_1t_2 - t_2 \end{bmatrix}$$

$V_1(G)$ = zero set of 2×2-minors

1st minor $=(1-t_3)(t_1-1)-t_1(1-t_3)(1-t_2t_3)=(1-t_3)(t_1-1-t_1+t_1t_2t_3)$

2nd minor $=(t_2-1)(1+t_3-t_2)$

3rd minor $=(t_1-1)(1-t_1t_2t_3)$

$$V_1(G)=\{t\in(\mathbb{K}^*)^n \mid t_1t_2t_3-1=0\}$$
$$V_2(G)=\{1\}$$

Problem 4:  $\}$ n lines

$G\cong F_{n-1}\times F_1$

$$V_1=\{t\in(\mathbb{K}^*)^n \mid t_1\cdots t_n-1=0\}$$
$$\parallel$$
$$V_2=\cdots=V_{n-1} \qquad V_n=\{1\}$$

Problem 5

$$V_d(\mathcal{A})=\left\{t\in(\mathbb{K}^*)^n \;\middle|\; \begin{array}{l}(t_1,\dots,t_{n-1})\in V_d(d\mathcal{A})\\ \&\ t_1\cdots t_n-1=0\end{array}\right\}$$

General description cv's of $\mathcal{A}$'s (for $\mathbb{K}=\mathbb{C}$)

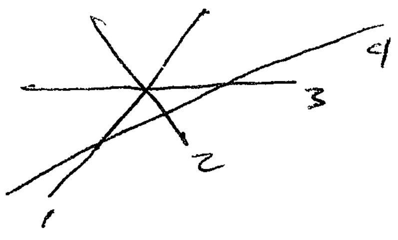
$$V_d(\mathcal{A})=\bigcup_{i=1}^{s}\rho_i T_i$$

where $T_i=(\mathbb{C}^*)^{r_i}$ is a subtorus of $T=(\mathbb{C}^*)^n$

ρ_i has finite order $(\rho_i^{k_i}=1)$

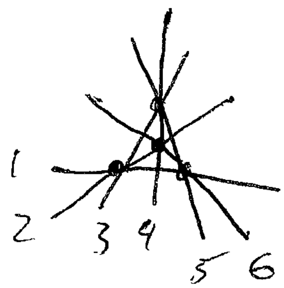
This follows from work of - Green-L.
Simpson
- Arapura

More examples



$$V_1 = \{ t \in (\mathbb{C}^*)^4 \mid t_1 t_2 t_3 = 1, \ t_4 = 1 \}$$

is a 2-dim. Torus



$$V_1 = \{ t \in (\mathbb{C}^*)^6 \mid t_1 t_2 t_3 = t_4 = t_5 = t_6 = 1 \} \cup$$

$$\{ t \mid t_2 t_4 t_6 = t_1 = t_3 = t_5 = 1 \} \cup$$

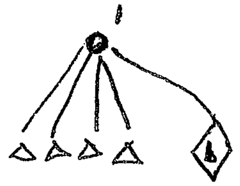
$$\{ t \mid t_3 t_4 t_5 = t_1 = t_2 = t_6 = 1 \} \cup$$

$$\{ t \mid t_1 t_5 t_6 = t_2 = t_3 = t_4 = 1 \} \cup$$

$$\{ t \mid t_1 = t_4, \ t_2 = t_5, \ t_3 = t_6, \ t_1 \cdots t_6 = 1 \}$$

$$\cancel{P_1 \cup P_2 \cup P_3} = T_1 \cup T_2 \cup T_3 \cup T$$

V_2



Connection to resonance varieties

$$R_d(G_*) = \{ a \in \mathbb{K}^n \mid \dim_{\mathbb{K}} H^1(H^*(G, \mathbb{K}), \cdot a) \geq d \}$$

$$= \{ a \in \mathbb{K}^n \mid \operatorname{rank}_{\mathbb{K}} M^{lin}(a) < n-d \}$$

where M^{lin} = linearized Alex. matrix

$t_i \longrightarrow 1 - a_i$ then take linear terms

$t_i^{-1} \longrightarrow 1 + a_i + a_i^2 + \cdots$

This is now a matrix of linear form

e.g. $\mathcal{A} =$ 

$$M = \begin{bmatrix} 1 - t_3 & t_1(1-t_3) & t_1 t_2 - 1 \\ 1 - t_2 t_3 & t_1 - 1 & t_2(t_1 - 1) \end{bmatrix}$$

$$M^{lin} = \begin{bmatrix} a_3 & a_3 & -a_1 - a_2 \\ a_2 + a_3 & -a_1 & -a_1 \end{bmatrix}$$

$$R_1 = \{ a \in \mathbb{K}^3 \mid a_1 + a_2 + a_3 = 0 \}$$

Theorem:

tangent cone

$$T_1(V_d(\mathcal{A}, \mathbb{C})) = R_d(\mathcal{A}, \mathbb{C})$$

in particular $R_d(\mathcal{A}, \mathbb{C})$ is a union of subspaces in $\mathbb{C}^n$.

Comments: • $T_1(V_d^i(\mathcal{A}, \mathbb{C})) = R_d^i(\mathcal{A}, \mathbb{C})$

- $T_1(V_d(G, \mathbb{C})) \subseteq R_d(G, \mathbb{C})$ $\neq$ in general
- $R_d(\mathcal{A}, \mathbb{K})$ need not be linear if char $k > 0$
-