Lecture 9 — Suciu

Fox Calculus & Alexander Invariants

G grp $\quad \mathbb{Z}G = \{ \sum_{g\in G} n_g g \mid n_g \in \mathbb{Z} \}$ finite

with multiplication given by the group

$\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ augmentation map
$g \mapsto 1$

$I = IG = \ker(\mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z})$ augm. ideal

$= \operatorname{span}(g-1 \mid g \in G \setminus \{1\})$

Assume G is finitely generated by $x_1, \dots, x_n$

Then $IG = \operatorname{span}(x_i - 1 \mid i = 1, \dots, n)$

Is it a free $\mathbb{Z}G$-mod?

Answer: iff $G = F_n$ (Stallings)

We have $IF_n \cong (\mathbb{Z}F_n)^n$

top. hint $\tilde{C}_*(\overset{n}{\vee} S^1) : 0 \to \underset{C_1}{(\mathbb{Z}F_n)^n} \xrightarrow[\begin{pmatrix} x_1-1 \\ \vdots \\ x_n-1 \end{pmatrix}]{\partial_1} \underset{C_0}{\mathbb{Z}F_n} \xrightarrow{\varepsilon} \mathbb{Z} \to 0$

snowflake $\qquad \operatorname{im}\partial_1 = IF_n$

Let $w \in F_n$

$$w - 1 = \sum_{i=1}^{n} \frac{\partial w}{\partial x_i}(x_i - 1)$$

coeff.s of $w-1$ in that basis expansion called Fox derivatives of w.

Extend linearly to

$$\partial/\partial x_i : \mathbb{Z}F_n \longrightarrow \mathbb{Z}F_n$$

uniquely characterized by two rules:

- $\dfrac{\partial x_j}{\partial x_i} = \delta_{ij}$

- $\dfrac{\partial (uv)}{\partial x_i} = \dfrac{\partial u}{\partial x_i}\,\varepsilon(v) + u \cdot \dfrac{\partial v}{\partial x_i}$

Additional properties

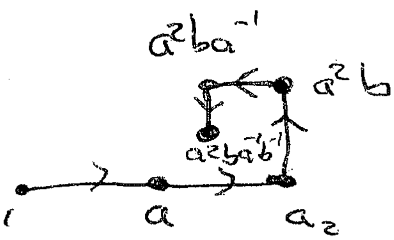
- $\dfrac{\partial 1}{\partial x_i} = 0$

- $\dfrac{\partial (x_i^{-1})}{\partial x_i} = -x^{-1}$

eg $\quad w = a^2 b a^{-1} b^{-1}$

$$\frac{\partial w}{\partial a} = 1 + a - a^2 b a^{-1}$$

$$\frac{\partial w}{\partial b} = a^2 - a^2 b a^{-1} b^{-1}$$



Resolution of $\mathbb{Z}$ over $\mathbb{Z}G$

Assume $\quad G = (x_1, \dots, x_n \mid y_1, \dots, y_m) \overset{\varphi}{\twoheadleftarrow} F_n = (x_1, \dots$

$$\tilde{C}_*\left(\widetilde{K(G,1)}\right) \cdots \to (\mathbb{Z}G)^m \xrightarrow[\;J_G\;]{\partial_2} (\mathbb{Z}G)^n \xrightarrow[\begin{pmatrix} x_1 - 1 \\ \vdots \\ x_n - 1 \end{pmatrix}]{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where J_G is the Fox Jacobian of G

$$J_G = \left(\frac{\partial y_i}{\partial x_j}\right) \in \mathrm{Mat}_{n \times m}(\mathbb{Z}G)$$

Eg $G = (a,b \mid aba^{-1}b^{-1} = 1) = \mathbb{Z}^2$

$K(G,1) = T^2$

$$\tilde{C}_*(\tilde{T}^2) \qquad \mathbb{Z}\mathbb{Z}^2 \xrightarrow{(1-b \ \ a-1)} (\mathbb{Z}\mathbb{Z}^2)^2 \xrightarrow{\binom{a-1}{b-1}} \mathbb{Z}\mathbb{Z}^2 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

Exercise (optional) Compute a free $\mathbb{Z}\mathbb{Z}^n$- resolution of $\mathbb{Z}$ (Koszul reso.)

Alexander Module/Invariant

X top space (path-conn.)

$G = \pi_1(X, *)$ $\qquad G' = [G,G]$

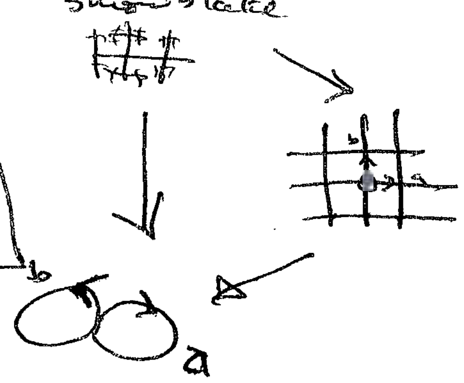
$$1 \to G' \to G \to G/G' \to 1$$

$$G/G' = G^{ab} = H_1(X)$$

eg. $X = M(\mathcal{L})$ $\quad H_1(X) = \mathbb{Z}^n$; $|\mathcal{L}| = n$

Let $G^{ab} \curvearrowright \tilde{X} \xrightarrow{p} X$ be the universal abel. cover

eg. snowflake



$A = H_1(\tilde{X}, p^{-1}(*)) = \mathbb{Z}G^{ab} \otimes_{\mathbb{Z}G} IG$ $\boxed{\text{Alex. mod.}}$

$B = H_1(\tilde{X}) = G'/G''$ $\boxed{\text{Alex. Invariant}}$

veiwed as modules over $\mathbb{Z}G^{ab}$

Exercise $0 \to B \to A \to I \to 0$ (Crowell exact seq.)

Now assume

$$\begin{array}{ccc} G = (x_1,\dots,x_n \mid r_1,\dots,r_m) & & x_i \\ \downarrow{\scriptstyle ab} & & \downarrow \\ G^{ab} = \mathbb{Z}^n = (t_1,\dots,t_n \mid [t_i,t_j]=1) & & t_i \end{array}$$

identify $\mathbb{Z}\mathbb{Z}^n \cong \mathbb{Z}[t_1^{\pm 1},\dots,t_n^{\pm 1}] = \Lambda$

(i.e. $r_i \in F_n'$) (eg. $G = \pi_1(\mathbb{C}^2 \setminus \cup \mathcal{A})$)

$$\begin{tikzcd}
C_*(\tilde{X}): & & \Lambda^m \arrow[r, "d_2 = J_G^{ab}"] \arrow[d, dashed, "\varphi"] & \Lambda^n \arrow[r, "d_1"] \arrow[d, equal] & \Lambda \arrow[r, "\varepsilon"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
\tilde{C}_*(\tilde{T}^n): \arrow[r] & \Lambda^{\binom{n}{3}} \arrow[r, "d_3"] & \Lambda^{\binom{n}{2}} \arrow[r, "d_2"] & \Lambda^n \arrow[r, "d_1"] & \Lambda \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Then

$$A = \operatorname{coker}\left(J_G^{ab}\right) \quad \text{Alexander matrix}$$

$$B = \operatorname{coker}\begin{pmatrix} \varphi \\ d_3 \end{pmatrix} \quad \text{mapping cone } \Lambda^m \oplus \Lambda^{\binom{n}{3}} \xrightarrow{\varphi,\, d_3}$$

<u>Exercise</u> (optional) $\varphi_3 = \operatorname{rank}(B/IB)$