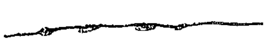
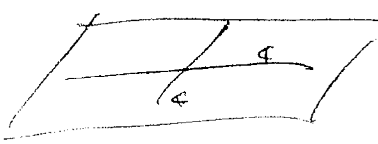
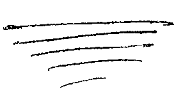
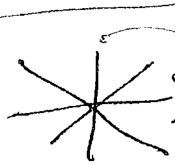
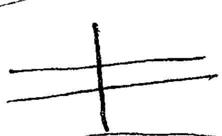
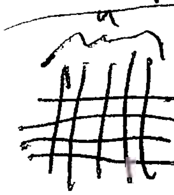
Lecture 3 Suciu

Fundamental Groups of Arrangements

$\mathcal{A} = \{H_1, \dots, H_n\}$ hyp. arr. in $\mathbb{C}^\ell$

$M = M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup H_i$, $G = G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$.
(path connected)

Examples

	$\mathcal{A}$	$M(\mathcal{A})$	$G(\mathcal{A})$
$\ell=1$	$\bullet$ $\mathbb{C}$ 0	$\simeq S^1$	$\mathbb{Z}$
	 n pts	$\simeq \vee S^1$	F_n
$\ell=2$	 $\mathbb{C}^2$	$\mathbb{C}^* \times \mathbb{C} \simeq S^1$	$\mathbb{Z}$
	 $\}$ n lines	$\mathbb{C} \setminus n$ pts	F_n
	 $\}$ n lines	$S^2 \setminus (n \text{ pts}) \times S^1$	$F_{n-1} \times F_1$
		$\mathbb{C}^* \times (\mathbb{C} \setminus 2\text{pts})$	$F_2 \times F_1$
	 n, $\}$ m	$\mathbb{C} \setminus (m \text{ pts}) \times \mathbb{C} \setminus (n \text{ pts})$	$F_m \times F_n$

ℓ $\mathcal{A}_\ell = \{H_{ij}\}_{1 \le i < j \le \ell}$ $\quad F_\ell(\mathbb{C})$ $\quad P_\ell = F_{\ell-1} \rtimes_{\alpha_{\ell-1}} F_{\ell-2} \rtimes \cdots \rtimes_{\alpha_2} F_1$

$H_{ij} = \ker\{z_i - z_j\}$ $\quad$ conf. space of

braid arr.

$n = \binom{\ell}{2}$ $\quad \alpha_\ell : P_\ell \to \mathrm{Aut}(F_\ell)$

By taking a generic 2-slice, we can assume $\ell = 2$.

$$\mathcal{A} \text{ in } \mathbb{C}^\ell \xrightarrow{\text{slice}} \mathcal{A}_1' \text{ in } \mathbb{C}^2$$

set of hyperplanes $\quad$ lines

By a theorem of Zariski & Lefschetz, as proved by Hamm & Lê

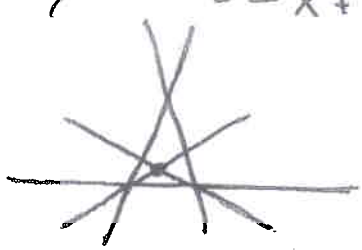
$$\pi_1(M(\mathcal{A})) = \pi_1(M(\mathcal{A}'))$$

e.g.: $\mathcal{A}_4$ braid arr. in $\mathbb{C}^4$

$\mathcal{A}_4 = \mathcal{A} \times \mathbb{C}$ where $\mathcal{A}$ is given by $Q(\mathcal{A}) = xyz(x-y)(x-z)(y-$

take a generic section

say $z = x + 3y + 59$ get



So, now take $\mathcal{A} = \{H_1, \dots, H_n\}$ affine arr. of lines in $\mathbb{C}^2$. By making a change of coord. we may assume

$p : \mathbb{C}^2 \longrightarrow \mathbb{C}$

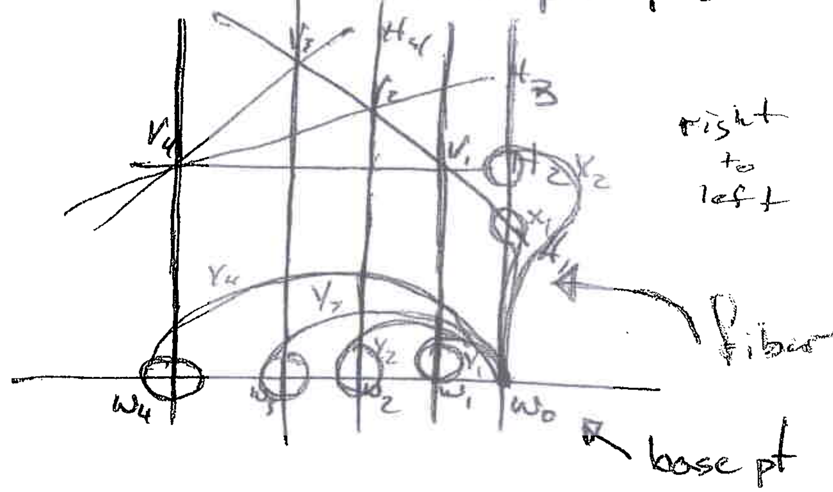
$p(x,y) = x$ is generic w.r.t. $\mathcal{A}$.

$\Longleftrightarrow$ $p^{-1}(x)$ does not contain a line H_i and not more than one multiple pt.

$\mathbb{C}^2 = (x,y)$

$\downarrow p$

$\mathbb{C} = (x)$



$v_1, \dots, v_s$ multiple pts of $\mathcal{A}$

$w_k = p(v_k)$

$L_k = p^{-1}(w_k)$

$\hat{\mathcal{A}} = \{H_1, \dots, H_n, L_1, \dots, L_s\}$

$M(\hat{\mathcal{A}})$ complement

$$\begin{array}{ccc} \mathbb{C} \setminus (n\text{ pts}) & \longrightarrow & M(\hat{\mathcal{A}}) \\ & & \downarrow \\ & & \mathbb{C} \setminus (s\text{ pts}) \end{array}$$

is a bundle

To show this is a bundle we use the diagram fiber

$$\begin{tikzcd}
\mathbb{C}\setminus(n\text{ pts}) \arrow[rr, equal] \arrow[d] & & \mathbb{C}\setminus(n\text{ pts}) \arrow[d] \\
\mathcal{M}(\mathcal{A}_{n+1}) = F_{n+1}(\mathbb{C}) \arrow[d] & & \mathcal{M}(\hat{\mathcal{A}}) \arrow[ll] \arrow[d] \\
\mathcal{M}(\mathcal{A}_n) = F_n(\mathbb{C}) \arrow[u, bend left, "s_{n+1}"] & & \mathcal{M}\,\mathbb{C}\setminus(s\text{ pts}) \arrow[ll, "b"'] \arrow[u, bend right, "\sigma"']
\end{tikzcd}$$

section

where $b = (b_1, \dots, b_n)$ where

$Q(\mathcal{A}) = (y - b_1(x)) \cdots (y - b_n(x))$ $\qquad b_i(x) = m_i x + c_i$

Complement is total space of the pullback bundle

$$\begin{array}{ccccc}
b_* : & \pi_1(\mathbb{C}\setminus(s\text{ pts})) & \longrightarrow & \pi_1(\mathcal{M}(\mathcal{A}_n)) & \\
\| & \| & & \| & \\
B : & F_s & \longrightarrow & P_n & B = (B_1, \dots, B_s) \\
 & & & & B_k \in P_n
\end{array}$$

the induced homom

is called the <u>braid</u> <u>monodromy</u> of $\mathcal{A}$.

L.e.s. in π_* for the bundle

$$1 \to \underset{\substack{\| \\ F_n}}{\pi_1(\mathbb{C}\setminus(n\text{ pts}))} \to \pi_1(\mathcal{M}(\hat{\mathcal{A}})) \xrightarrow{\;\sigma_*\;} \underset{\substack{\| \\ F_s}}{\pi_1(\mathbb{C}\setminus(s\text{ pts}))} \to 1$$

thus $\pi_1(\mathcal{M}(\hat{\mathcal{A}})) = F_n \rtimes_B F_s$

where $B : F_s \to P_n \overset{\alpha_n}{\hookrightarrow} \mathrm{Aut}(F_n)$ $\qquad = \langle x_1, \dots, x_n, y_1, \dots, y_s \mid y_k^{-1} x_i y_k = B_k(x_i) \rangle$

Filling in $\bigcup L_k \setminus V_k$ gives back $M(\mathcal{A})$

(i.e. $M(\mathcal{A}) = M(\tilde{\mathcal{A}}) \cup \bigcup_{k=1}^{s} (L_k - V_k)$)

Use van Kampen to get

$$\boxed{\pi_1(M(\mathcal{A})) = \left(x_1, \ldots, x_n \;\middle|\; x_i = B_k(x_i) \right)}$$

$k = 1, \ldots, s$

where m_k = multiplicity of V_k $i = i_1, \ldots, i_{m_k - 1}$

if $V_k = H_{i_1} \cap \cdots \cap H_{i_{m_k}}$

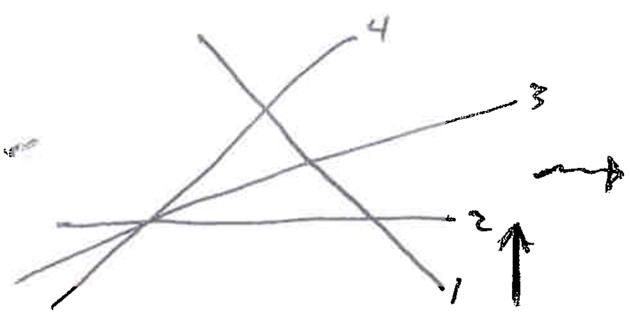
This is the minimal presentation for π_1. It carries the homotopy type of $M(\mathcal{A})$.

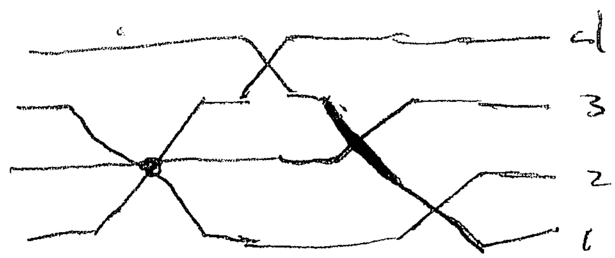
$M(\mathcal{A}) \simeq$ presentation 2-complex

Explicit way to find braid monodromy generators

Read them off a "wiring diagram" (in complexified real case)

"braiding wiring diagram" (in general)





$B_4 = A_{234}$ $B_3 = A_{14}$ $B_2 = A_{13}$ $B_1 = A_{12}$

where

where if $I = (i_1, \dots, i_r)$

$$A_I = A_{i_1 i_2} A_{i_1 i_3} A_{i_2 i_3} \cdots A_{i_1 i_r} \cdots A_{i_{r-1} i_r}$$

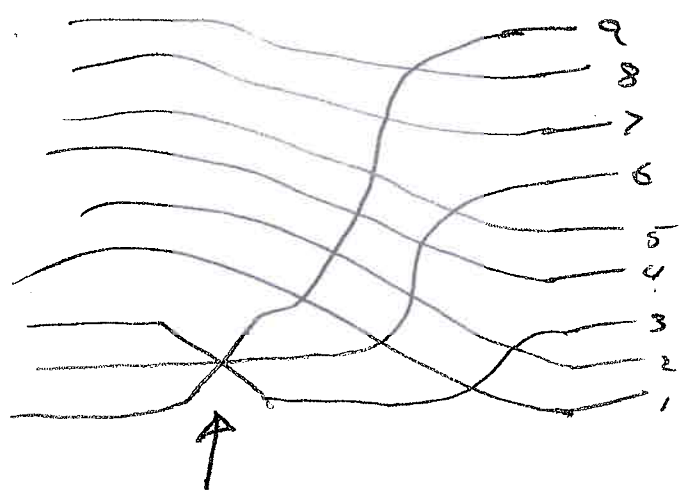
in general:

$$B_K = A_{I_K}^{\delta_K} \qquad \left(\text{notation } a^b = b^{-1} a b\right)$$

where $I_K = \{\text{indices of wires meeting at } V_K\}$

$$\delta_K = \prod_{i \in I_K} \prod_{j \in J_K} A_{ji}$$

where J_K = indices of middle wires that go above I_K at V_K and intersect the intersect wires



$$B = A_{369}^{A_{46} A_{56} A_{49} A_{59} A_{79} A_{89}}$$