Braids & Fundamental Grps of Arrangements

1. Braid Grps B_n
2. Pure Braids P_n
3. Braid Monodromy
4. Fundamental Gps

1.1 Braids (Artin 1920's)

Braid grp B_n = braids on n strings
with $\bullet$ = concatenation from top to bottom

- identity = trivial braid
- inverse = mirror image

Ex: $B_1 = 1$, $B_2 = \mathbb{Z}$, $B_3 = \dots$

1.2 Presentation:

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad 1 \le i \le n-2 \quad \text{and} \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| \ge 2 \rangle$$

where $\sigma_i =$

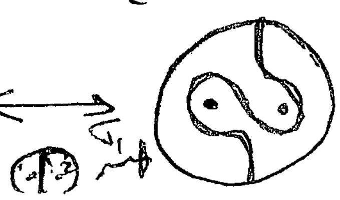
1.3 Dynamical interpretation

braid $\leftrightarrow$ orientation preserving diffeomorphism of D^2, preserving pt. wise ∂D^2 and permuting n marked pts

braid equivalence $\leftrightarrow$ isotopy

eg.

$\longleftrightarrow$

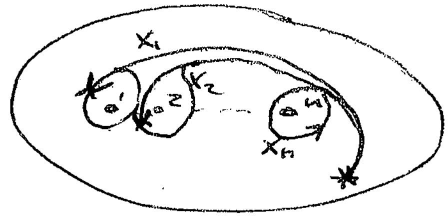


upshot: $B_n = \mathcal{M}^n_{0,1}$
mapping class grps

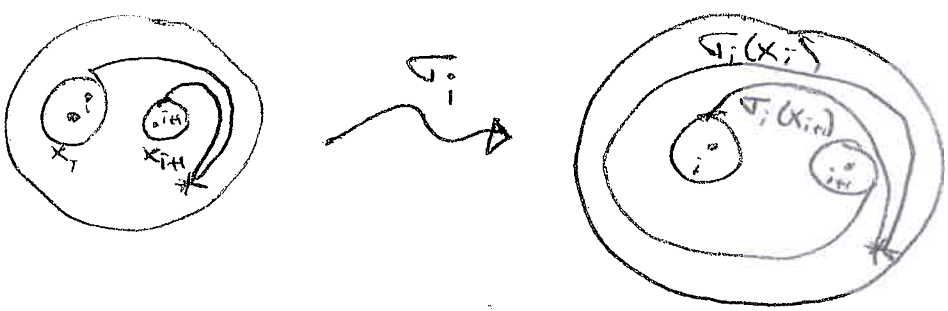
1.4 Artin Representation

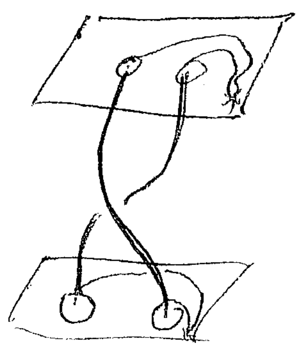
$$\alpha_n : B_n \hookrightarrow \mathrm{Aut}(F_n) \quad \text{injective homomorphism}$$

$$\beta \longmapsto \beta_* : \underset{F_n}{\overset{\shortparallel}{\pi_1(D^2 \setminus n\text{ pts})}} \longrightarrow \underset{F_n}{\overset{\shortparallel}{\pi_1(D \setminus n\text{ pts})}}$$



Compute $\quad \alpha_n(\sigma_i) = (\sigma_i)_*$





$$\sigma_i : \begin{cases} x_i \longmapsto x_i x_{i+1} x_i^{-1} \\ x_{i+1} \longmapsto x_i \end{cases}$$

$$\sigma_i^{-1} : \begin{cases} x_i \longmapsto x_{i+1} \\ x_{i+1} \longmapsto x_{i+1}^{-1} x_i x_{i+1} \end{cases}$$

boundary preserved

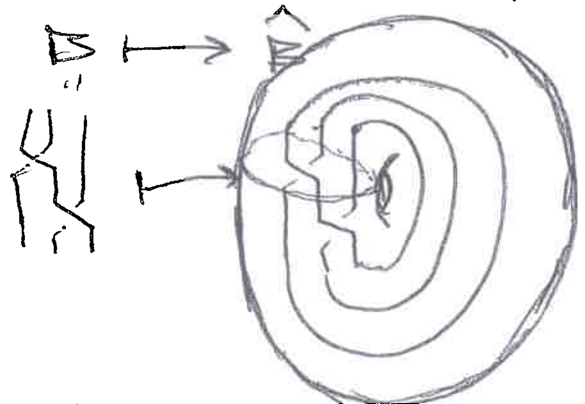
Remark: B_n is the subgp of $\mathrm{Aut}(F_n)$ consisting of $\varphi : F_n \longrightarrow F_n$ s.t.

- $\varphi(x_i) = w_i x_{\pi(i)} w_i^{-1}$
- $\varphi(x_1 \cdots x_n) = x_1 \cdots x_n$

1.5 Closed-up braids

Braids $\longrightarrow$ Links (of circles in S^3)

$\mathcal{B} \longmapsto \hat{\mathcal{B}}$



$\#$ components $=$ $\#$ cycles in $\overline{\mathcal{B}}$

Step 1: Trivial Bundle

$$D^2 \longrightarrow D^2 \times S^1 \xrightarrow{\quad} S^1$$

Now remove $\hat{\mathcal{B}}$ from $D^2 \times S^1$ compute $\pi_1(S^3 \setminus \hat{\mathcal{B}})$

$$D^2 \setminus \{n \text{ pts}\} \longrightarrow D^2 \times S^1 \setminus \hat{\mathcal{B}} \underset{s}{\overset{p}{\rightleftarrows}} S^1$$

s (section)

(get non-trivial bundle with "monodromy" $\mathcal{B}$ i.e.

$$D^2 \times S^1 \setminus \hat{\mathcal{B}} = (D^2 \setminus n \text{ pts}) \times I \Big/ (x,0) \sim (\beta(x), 1)$$

Review: $F \to E \to B$

$$\cdots \to \pi_i(F) \to \pi_i(E) \longrightarrow \pi_i(B) \to \pi_{i-1}(F) \to \cdots$$

l.e.s. in π_*

$$(S^1) \to \pi_1(D^2 \setminus \{n \text{ pts}\}) \longrightarrow \pi_1(D^2 \times S^1 \setminus \hat{\mathcal{B}}) \underset{s_*}{\rightleftarrows} \pi_1(S^1) \longrightarrow 1$$

$$1 \qquad\qquad F_n \qquad\qquad\qquad\qquad\qquad\qquad \mathbb{Z}$$

Since there is a section we have a splitting of the ses.

$$\pi_1(D^2 \times S^1 \setminus \hat{B}) = F_n \rtimes_B \mathbb{Z}$$

$$= \langle x_1, \dots, x_n, t \mid t^{-1} x_i t = B(x_i) \rangle_{i=1,\dots,n}$$

<u>Step 2</u>: $S^3 = D^2 \times S^1 \bigcup_{S^1 \times S^1} S^1 \times D^2$

So, $S^3 \setminus \hat{B} = (D^2 \times S^1 \setminus \hat{B}) \bigcup_{S^1 \times S^1} S^1 \times D^2$

Now, use van Kampen

$$\begin{tikzcd}
 & \pi_1(S^1 \times D^2) \arrow[dr] & \\
\pi_1(S^1 \times S^1) \arrow[ur] \arrow[dr] & & \pi_1(S^3 \setminus \hat{B}) \\
 & \pi_1(D^2 \times S^1 \setminus \hat{B}) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \overset{a}{\mathbb{Z}} \arrow[dr] & \\
\underset{a}{\mathbb{Z}} \times \underset{b}{\mathbb{Z}} \arrow[ur, "a \mapsto a,\ b \mapsto 1"] \arrow[dr] & & G \\
 & F_n \rtimes \mathbb{Z} \arrow[ur] &
\end{tikzcd}$$

$a \longmapsto x_1 \cdots x_n$

$b \longmapsto t$

upshot

$$\pi_1(S^3 \setminus \hat{B}) = \langle x_1, \dots, x_n \mid x_i = B(x_i) \rangle_{i=1,\dots,n}$$

<u>Example</u>: (Hopf link)

Full twist on n strands $B_1 = \sigma_1^2$

$B_2 = (\sigma_1 \sigma_2 \sigma_1)^2$

$\hat{B}_n = \mathrm{Hopf}_n = L_{n,n}$

Compute $B(x_i) = x_i^{x_1 \cdots x_n}$

Hence $\pi_1(S^3 \setminus \hat{B} = L_{n,n}) = (x_1, \dots, x_n \mid x_i = x_i^{x_1 \cdots x_n})$

$= (x_1, \dots, x_n \mid x_1 \cdots x_n \text{ central})$

2. Pure Braid Grps

2.1 Generators for $P_n = \ker(B_n \to S_n)$

$A_{ij} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$ $\quad 1 \le i < j \le n$

$A_{12} = \sigma_1^2 =$ 

$A_{13} = \sigma_1 \sigma_2^2 \sigma_1^{-1} =$ 

Relations: complicated

example: (exercise!) $\Delta_n^2 = (A_{12})(A_{13} A_{23})(A_{14} A_{24} A_{34}) \cdots (A_{1n} A_{2n} \cdots A_{n-1\,n})$

2.2 Topological interpretation

X - top. space

$F_n(X) = \{(x_1, \dots, x_n) \in X \times \cdots \times X \mid x_i \neq x_j \ \forall i \neq j\}$

configuration space of n ordered pts in X.

S_n acts on $F_n(X)$ by permuting coordinates

$C_n(X) = F_n(X)/S_n$ = conf. space of n unordered pts in X

Have $S_n \to F_n(X)$
$$\downarrow$$
$C_n(X)$ regular cover

get

$$(*) \quad 1 \to \pi_1(F_n(X)) \to \pi_1(C_n(X)) \to S_n \to 1$$

Now take $X = \mathbb{C}$

$F_n(\mathbb{C})$ = complement of braid arr.

$$\mathcal{A}_n = \{H_{ij}\}_{1 \le i < j \le n} \qquad H_{ij} = \{z_i - z_j = 0\}$$

$$\downarrow$$

$C_n(\mathbb{C})$ = discriminant space of monic polynomials of deg. n w/ no repeated roots

Notice that $(*)$ becomes

$$\begin{array}{ccccccccc} 1 \to & \pi_1(F_n(\mathbb{C})) & \to & \pi_1(C_n(\mathbb{C})) & \to & S_n & \to 1 \\ & \parallel & & \parallel & & & \\ & P_n & & B_n & & & (\text{Fox}) \end{array}$$

reason (braid B) $\longleftrightarrow$ ~~path~~ loop in conf. space $C_n(\mathbb{C})$

braid equivalence $\longleftrightarrow$ homotopy of loops

2.3 Semidirect product structure on P_n

$\mathbb{C} \to \mathbb{C}^n \xrightarrow{p_n} \mathbb{C}^{n-1}$ trivial bundle

$(z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-1})$

Restrict to conf. spaces

$\mathbb{C} \setminus (n-1 \text{ pts}) \longrightarrow F_n(\mathbb{C}) \xrightarrow{p_n} F_{n-1}(\mathbb{C})$ (non-trivial) bundle w/ fiber $\mathbb{C} \setminus n-1$ pts

$p_n^{-1}(z_1, \dots, z_{n-1}) = \{(z_1, \dots, z_n) \mid z_n \neq z_1, \dots, z_n \neq$

Moreover p_n admits a section

Use these bundles to understand $F_n(\mathbb{C})$ and

$P_n = \pi_1(F_n(\mathbb{C}))$ inductively

$F_1(\mathbb{C}) = \mathbb{C}$

$F_2(\mathbb{C}) = \mathbb{C} \times \mathbb{C}^*$

$\vdots$

$$\begin{tikzcd}
F_n \arrow[r] & F_{n-1} \arrow[r] & \cdots \arrow[r] & F_2 \arrow[r] & F_1 = \mathbb{C} \\
\mathbb{C}_{n-1} \arrow[u] & \mathbb{C}_{n-2} \arrow[u] & & \mathbb{C}_1 \arrow[u] &
\end{tikzcd}$$

get
- $F_n(\mathbb{C}) = K(P_n, 1)$
- $P_n = F_{n-1} \rtimes_{A_{n-1}} P_{n-1} = F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_2 \rtimes F_1$

e.g. $P_2 = F_1$

$P_3 = F_2 \rtimes F_1 \cong F_2 \times F_1$

$P_4 = F_3 \rtimes F_2 \rtimes F_1 \not\cong F_3 \times F_2 \times F_1$