MSRI Summer Graduate School on Hyperplane Arrangements and Applications

August 2-13, 2004

Homework Problems, third installment

1. Show $D(\mathcal{A}) = \{\theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A}\}$
2. Let $Q(\mathcal{A}) = xyz(x+y+z)$. Show that $\mathcal{A}$ is not free.
3. Show that every central arrangement in a 2-dimensional space is free.
4. Define the Euler derivation $\theta_E \in \mathrm{Der}_S$ by $\theta_E = \sum_{i=1}^{\ell} x_i \partial_i$. Show that θ_E can be a member of a basis for $D(\mathcal{A})$ for every nonempty central free arrangement.
5. Find a basis for $D(B_\ell)$.
6. Find a basis for $D(D_\ell)$.
7. Show that braid arrangement A_ℓ is inductively free with exponents $(1, 2, 3, \dots, \ell)$.
8. Prove the first part of Folkman's Theorem, that the atomic complex of the intersection lattice has nonzero homology only in the top dimension $\ell - 2$.
9. Prove that the relative atomic complex D_X is a differential graded algebra under the product
$$\sigma \cdot \tau = (-1)^{\mathrm{sgn}(\sigma,\tau)}\, \sigma \cup \tau,$$
where $\mathrm{sgn}(\sigma, \tau)$ is the sign of the permutation of (σ, τ) into increasing order.
10. Let A be the OS algebra of an arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$, and A' the OS algebra of the deletion $\mathcal{A}' = \mathcal{A} - \{H_n\}$ of $\mathcal{A}$. Show that the natural map $A' \to A$ is an injective algebra map.
11. Let A and A' be as above, and let A'' be the OS algebra of the restriction $\mathcal{A}'' = \{H_i \cap H_n \mid 1 \le i \le n-1\}$ of $\mathcal{A}$. The "residue map" $A \to A''$ is defined by $e_S \mapsto 0$ if $n \notin S$ and $e_S \mapsto e_{S-n}$ for $n \in S$.
    (a) Show that the residue map is well-defined.
    (b) Show that $0 \to A' \to A \to A''(-1) \to 0$ is an exact sequence of degree zero maps. (The notation $A''(-1)$ means A'' with the grading shifted by 1: $A''(-1)^k = (A'')^{k-1}$.)
12. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement and $\alpha_k : \mathbb{C}^\ell \to C$ with $H_k = \ker \alpha_k$. Show that the one-forms $\frac{d\alpha_k}{\alpha_k}$ satisfy the Orlik-Solomon relations.

13. Define $\partial : A^p \to A^{p-1}$ by

$$\partial e_S = \sum_j (-1)^{j-1} e_{S-i_j}$$

for $S = \{i_1, \dots i_p\}$.

(a) Show that ∂ is well-defined.

(b) Let $a = \sum_{i=1}^n \lambda_i e_i \in A^1$, and denote also by a the map $A^p \to A^{p+1}$ given by multiplication by a. Show that $\partial a + a\partial : A^p \to A^p$ is given by multiplication by the scalar $\sum_{i=1}^n \lambda_i$.

14. Let G be a finitely-presented group, $\mathbb{Z}[G]$ its integral group ring, and $I \subset \mathbb{Z}[G]$ the augmentation ideal. Let B be the Alexander module and A the Alexander invariant of G. (Recall, B is the first homology $H_1(\tilde{X})$ of the universal abelian cover $\tilde{X}$ of $X = K(G, 1)$, and A is the relative homology $H_1(\tilde{X}, p^{-1}(x_0))$, considered as $\mathbb{Z}[G]$-modules.) Show there is a short exact sequence

$$0 \to B \to A \to I \to 0.$$

15. Let ϕ_3 be the rank of the third factor $G^{(3)}/G^{(4)}$ in the lower central series of G. Show that ϕ_3 is the rank of the $\mathbb{Z}[G]$-module B/IB.
(Hint: First show ϕ_3 is the rank of third quotient in the lower central series of G/G''.)