MSRI Summer Graduate School on
Hyperplane Arrangements and Applications

August 2-13, 2004

Homework Problems, second installment

1. (a) Show that the OS ideal for the rank-three braid arrangement with defining polynomial $Q(\mathcal{A}) = xyz(x-y)(x-z)(y-z)$ is generated by quadratic relations.

   (b) Find an arrangement for which the OS ideal is not generated by quadratic relations.

2. Show that e_S is in the OS ideal if S is dependent.

3. Show that the grading of the exterior algebra E by L given by $E_X = \langle e_S \mid \bigvee S = X \rangle$ for $X \in L$ induces a well-defined grading of the OS algebra A.

4. Show the natural map $A(\mathcal{A}_X) \to A(\mathcal{A})$ is injective for every $X \in L$.

5. Show that the atomic complex of L (whose simplices are sets of atoms whose join not equal to 1_L) is homotopy equivalent to the flag complex of $L - \{0_L, 1_L\}$.

6. Let $\mathcal{A}$ be the braid arrangement in $\mathbb{R}^{\ell+1}$:

$$\mathcal{A} = \{\ker(x_i - x_j) \mid 1 \le i < j \le \ell + 1\}.$$

   Find $T := T(\mathcal{A})$ and $\mu(T)$ when $\ell = 3$. (Remark. If you set

$$x = x_1 - x_2, \quad y = x_1 - x_3, \quad z = x_1 - x_4$$

   then $\mathcal{A} = \{\ker(x), \ker(y), \ker(z), \ker(x-y), \ker(y-z), \ker(x-z)\}$

7. Prove that $W(\mathcal{A})$ is a finite reflection group if $\mathcal{A}$ is a reflection arrangement.

8. Which is the more obvious?

   (a) $\mathcal{A}(W(\mathcal{A})) = \mathcal{A}$ for any reflection arrangement $\mathcal{A}$,

   (b) $W(\mathcal{A}(W)) = W$ for any finite reflection group W

   Prove the more obvious one and discuss the other one.

9. Let $\mathcal{A} = A_\ell$.

   (a) Show that $\mathcal{A}$ is a reflection arrangement.

   (b) Find $|\mathcal{A}|$ and $|W(\mathcal{A})|$.

   (c) Describe the group $W(\mathcal{A})$ or find a group isomorphic to $W(\mathcal{A})$.

10. Let $\mathcal{A} = B_\ell$.

   (a) Show that $\mathcal{A}$ is a reflection arrangement.

   (b) Find $|\mathcal{A}|$ and $|W(\mathcal{A})|$.

   (c) Describe the group $W(\mathcal{A})$ or find a group isomorphic to $W(\mathcal{A})$.

11. Let $\mathcal{A} = D_\ell$.

   (a) Show that $\mathcal{A}$ is a reflection arrangement.

   (b) Find $|\mathcal{A}|$ and $|W(\mathcal{A})|$.

   (c) Describe the group $W(\mathcal{A})$ or find a group isomorphic to $W(\mathcal{A})$.

12. Find a system of simple roots and the Coxeter diagram of B_3.

13. Let $\mathcal{A}$ be a reflection arrangement. Show $|\mu(X)| = |\{w \in W \mid \mathrm{Fix}(w) = X\}|$ for $X \in L(\mathcal{A})$.

14. Let V be an ℓ-dimensional vector space with a basis e_i $(1 \le i \le \ell)$. Define the inner product on V by

$$(e_i, e_i) = 4/9, \ \ (e_i, e_j) = -1/18 \ (i \neq j)$$

   Let

$$\begin{aligned}\mathcal{A}_\ell = \{&\ker(x_i - x_j) \ (1 \le i < j \le \ell), \\ &\ker(x_i + x_j + x_k) \ (1 \le i < j < k \le \ell), \\ &\ker(x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} + x_{i_5} + x_{i_6}) \ (1 \le i_1 < i_2 < i_3 < i_4 < i_5 < i_6 \le \ell)\}.\end{aligned}$$

   It is known that $\Delta := \{e_i - e_{i+1} \ (1 \le i < \ell), e_{\ell-2} + e_{\ell-1} + e_\ell\}$ is a system of simple roots. Let $E_6 = \mathcal{A}_6$ and $E_7 = \mathcal{A}_7$.

   (a) Find $|E_6|$ and $|E_7|$.

   (b) Find the Coxeter diagrams of E_6 and E_7.

   (c) Show E_6 and E_7 are reflection arrangement.