MSRI Summer Graduate School on Hyperplane Arrangements and Applications

August 2-13, 2004

Homework Problems, first installment

1. Derive the Randell relations from the Wirtinger presentation of an (n, n)-torus link, for $n = 4$. Show that the ensuing presentation of $G = \pi_1(S^3 - L_{4,4})$ is equivalent to the Artin presentation, and that $G \cong F_3 \times F_1$.

2. Comb the full twist on n strings, for $n = 3$ and $n = 4$. More specifically, if

$$\Delta_n := \sigma_1\sigma_2\sigma_1 \cdots \sigma_{n-1} \cdots \sigma_1 \in B_n$$

   is the half-twist on n strings, show that

$$\Delta_n^2 = A_{12}A_{13}A_{23} \cdots A_{1n} \cdots A_{n-1n} \in P_n.$$

3. Let $M(\mathcal{A}_n) = F_n(C)$ be the complement of the braid arrangement (or, the configuration space of n ordered points in $\mathbb{C}$). Recall the bundle map

$$p_n \colon M(\mathcal{A}_n) \to M(\mathcal{A}_{n-1}), \quad (z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-1}).$$

   Find a section $s_n \colon M(\mathcal{A}_{n-1}) \to M(\mathcal{A}_n)$ to p_n, for $n = 3$.

4. Show that the pure braid group P_4 is not isomorphic to the direct product $F_3 \times F_2 \times F_1$.
   [Hint: Use `GAP` to compute invariants that distinguish the two groups.]

5. Use the fact that the cohomology of the complement M is torsion-free and generated by degree-one classes to prove that the Hurewicz homomorphism $\phi_k \colon \pi_k(M) \to H_k(M)$ is trivial for $k \geq 2$.
   [Hints: Show that the mapping on k^{th} cohomology induced by the universal covering $p \colon \tilde{M} \to M$ is trivial. Apply the Universal Coefficient Theorem to carry this over to homology. Lift ϕ_k to $\tilde{M}$ and use the long exact homotopy sequence.]

6. (a) Describe explicitly the Huisman model for the (projective image of) the rank-three braid arrangement.
   (b) Show (in general, or in several examples) that the Huisman model has the right fundamental group.

7. Let $\mathcal{A}$ be the arrangement in $\mathbb{C}^2$ with defining polynomial $Q(\mathcal{A}) = xy(x + y + 1)$.
   (a) Show that the complement $M(\mathcal{A})$ has the homotopy type of the 2-skeleton of the 3-torus.
   (b) Show that $\pi_2(M(\mathcal{A}))$ is not 0.