Graduate Student Workshop in Hyperplane Arrangements

8/2/2004 — 8/13/2004

Contents

Notes on Arrangements

by MAX WAKEFIELD

References:

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" : " & Hyp Geo. Int.

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(Falk, Terao, ed.)

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Yuz. OS alg's in Top. & Alg.

Suciu : Fund. gps of arr's enumerative aspects
contemp math 276

Lecture 1 — Falk

Let $\mathcal{A} = \{L_1, \dots, L_n\}$ — arrangement of lines through 0 in $\mathbb{C}^2$, $M = M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{i=1}^{n} L_i$.

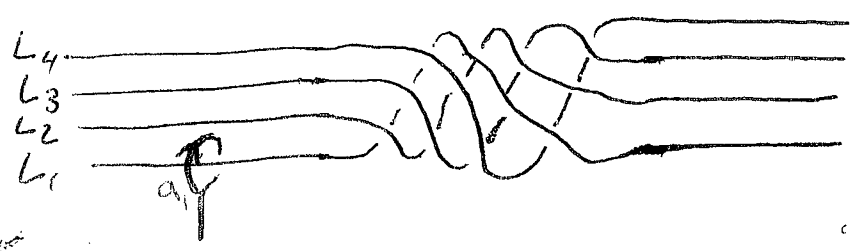
Then $M \simeq (M \cap S^3) \times (0,\infty) \simeq M \cap S^3$
$= S^3 \setminus \bigcup_{i=1}^{n} (L_i \cap S^3)$. The lines determine points in $\mathbb{CP}^1 \cong S^2$.

$L_i \cap S^3$ is a fiber of $h: S^3 \to \mathbb{CP}^1 = S^2$

$\{L_i \cap S^3\}$ are $(1,1)$ curves on tori in S^3

So, $\bigcup_{i=1}^{n} (L_i \cap S^3)$ is isotopic to an (n,n)-torus link $L_{n,n}$ in S^3 $\quad M \simeq S^3 - L_{n,n}$

ex: $n = 4$



Exercise: Calculate the Wirtinger presentation of

$$\pi_1(S^3 - L_{n,n}) = \langle a_1, \dots, a_n \mid \underbrace{a_n a_{n-1} \cdots a_2 a_1 = a_{n-1} a_{n-2} \cdots a_2 a_1 a_n = a_{n-2} \cdots a_1 a_n a_{n-1} = \cdots = a_1 a_n a_{n-1} \cdots a_2}_{\text{denoted } [a_n, \dots, a_n]} \rangle$$

The Randell relations

§2 $S^3 - L \longrightarrow \mathbb{CP}^1 - \{\text{one pt.}\} \cong D^2$

is a trivial bundle

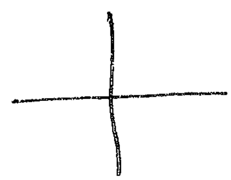
So, $S^3 \setminus L_{n,n} \longrightarrow \mathbb{CP}^1 - \{n \text{ pts}\}$ also trivial

So, $S^3 \setminus L_{n,n} \cong S^1 \times (S^2 \setminus n \text{ pts}) = D^2 - (n-1 \text{ pts})$

Then $\pi_1(S^3 - L_{n,n}) \cong F_1 \times F_{n-1}$

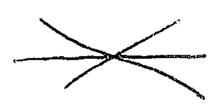
More precisely, $S^3 - L_{n,n} \simeq S^1 \times (\bigvee^{n-1} S^1)$

Ex: $n=2$



$M \simeq S^1 \times S^1$

$n=3$



$M \simeq S^1 \times (S^1 \vee S^1)$

= the union of two that meet along a $(1,1)$-curve

The Randell relations are equivalent to

$[a_n \cdots a_1, a_i] = 1 \quad \forall i$

§3 Let $\mathcal{A} = \{H_1, \dots, H_n\}$ an arrangement of linear* hyperplanes in $\mathbb{C}^\ell$

* $\mathcal{A}$ is central

$\pi : \mathbb{C}^\ell \setminus \{0\} \longrightarrow \mathbb{CP}^{\ell-1}$

$\pi(H_1)$

$=$

$\pi|_{\mathbb{C}^\ell \setminus H_1} : \mathbb{C}^\ell \setminus H_1 \longrightarrow \mathbb{CP}^{\ell-1} \setminus \overline{H}_1 \cong \mathbb{C}^\ell$

is a trivial bundle.

So, $\pi|_M : M \to \mathbb{CP}^\ell \setminus \bigcup_{i=1}^n H_i$ is trivial,

$$M \cong \mathbb{C}^* \times \left(\mathbb{CP}^{\ell-1} - \bigcup_{i=1}^n \bar{H}_i\right) \cong \mathbb{C}^{\ell-1} - \bigcup_{i=2}^n \left(\bar{H}_i \cap \mathbb{C}^{\ell-1}\right)$$

$\{\bar{H}_i \cap \mathbb{C}^{\ell-1} \mid 2 \le i \le n\}$ is an arrangement of $n-1$ affine hyperplanes in $\mathbb{C}^{\ell-1}$ called the <u>decone</u> of $\mathcal{A}$ written $d\mathcal{A}$.

<u>Thm</u>: $M(\mathcal{A}) \cong \mathbb{C}^* \times M(d\mathcal{A})$

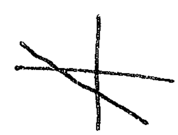
<u>Cor.</u>: $\chi(M(\mathcal{A})) = 0$

<u>Ex</u>: $\ell = 3$ $\mathcal{A}$ given by linear factors of $Q(\mathcal{A}) = xyz(x+y+z)$

$H_1 = \{z = 0\}$

$d\mathcal{A}$ given by $Q(d\mathcal{A}) = xy(x+y+1)$

is the projective image



The process is reversible: given an aff. arr. $\bar{\mathcal{A}}$ in $\mathbb{C}^{\ell-1}$ $\exists$ a central arr. $\mathcal{A}$ in $\mathbb{C}^\ell$ w/ $\bar{\mathcal{A}} = d\mathcal{A}$. $\mathcal{A}$ is called the <u>cone</u> of $\bar{\mathcal{A}}$, write $c\bar{\mathcal{A}}$.

§4 $\mathcal{A}$ is a central arr. in $\mathbb{C}^\ell$

$H_i = \ker(\alpha_i : \mathbb{C}^\ell \xrightarrow{\text{linear}} \mathbb{C})$. Say $\{H_{i_1}, \dots, H_{i_p}\}$ is <u>independent</u> if $\{\alpha_{i_1}, \dots, \alpha_{i_p}\}$ is linearly independent in $(\mathbb{C}^\ell)^*$

If $B \subseteq \mathcal{A}$ then $M(B) \overset{i}{\hookrightarrow} M(\mathcal{A})$ so

$$H^*(M(B)) \xrightarrow{i_*} H^*(M(\mathcal{A}))$$

If $B = \{H_{i_1}, \dots, H_{i_p}\}$ is independent then the linear map $\mathbb{C}^l \to \mathbb{C}^p$

$$x \longmapsto (\varphi_{i_1}(x), \dots, \varphi_{i_p}(x)) \quad \text{has kernel} = \bigcap_{k=1}^{p} H_{i_k} = X$$

and restricts to a homotopy equivalence

$$M(B) \longrightarrow (\mathbb{C}^*)^p$$

$$\left(M(B) \simeq (\mathbb{C}^*)^p \times X\right)$$ so, we get a

homom. $\Lambda(e_1, \dots, e_p) \cong H^*((\mathbb{C}^*)^p) \to H^*(M(\mathcal{A}))$

The images of these maps span $H^*(M(\mathcal{A}))$

Cor. $H^1(M)$ generates $H^*(M)$

Cor. (Randell) The Hurewicz homom.

$$\pi_k(M) \longrightarrow H_k(M) \quad \text{is trivial } \forall k \geq 2.$$

Proof: Exercise

§5 Randell's model for complexified real affine $\mathbb{Z}$-arr.

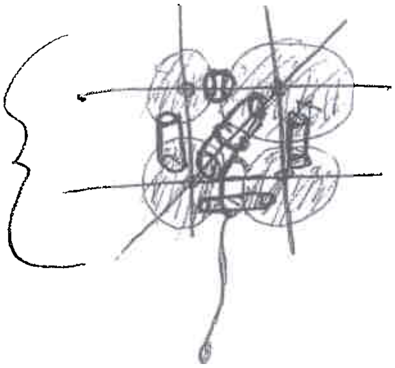
$\mathcal{A}$ is complexified real if the φ_i have real coeff

$\mathcal{A}$ = complexified real central $\mathbb{Z}$-arr.

$$\Rightarrow M(\mathcal{A}) \simeq \mathbb{C}^* \times M(d\mathcal{A}) = \mathbb{C}^* \times \overline{M}$$

complexified real affine $\mathbb{Z}$-arr.

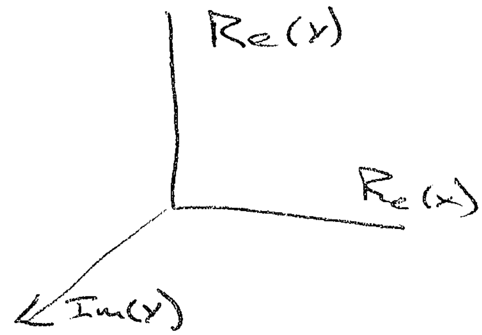
ex:

Γ 

For each intersection pt. X let $B_X = \overline{M} \cap B_{\mathbb{C}^2}(X, \epsilon)$

$B_X \simeq S^3 - L_{m,m}$ where m = mult. of X

$$\mathcal{M} \simeq \overline{M} \cap (\mathbb{R}^3 \times \overbrace{[-\delta', \delta']}^{\mathrm{Im}(y)})$$

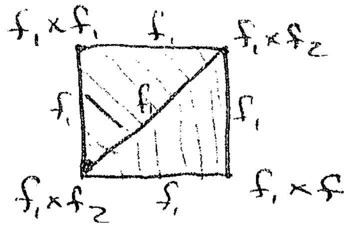


$$= (\mathbb{R}^3 - \Gamma) \cup \bigcup_X B_X$$

and $(\mathbb{R}^3 - \Gamma) \cap B_X = \mathbb{R}^3 - \ast$.

This yields the "tinker-toy" model of $\mathcal{M}$,

$\mathcal{M}$ = total space of a 2-complex of aspherical spaces



Huisman model for projective 2-arr.

$\mathcal{A}$ = central complexified real 3-arr.
$= \{H_1, \dots, H_n\}$

"Model for $\overline{M} = \mathbb{CP}^2 - \bigcup_{i=1}^{n} \overline{H_i}$."

Each $\overline{H_i}$ determines a line in ~~$\mathbb{CP}$~~ $\mathbb{RP}^2$

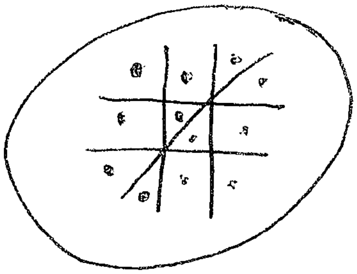
Let $\overline{P_i}$ = the point in $\mathbb{RP}^2$ dual to $\overline{H_i}$

Let $\pi : S^2 \to \mathbb{RP}^2$.

Let $\{P_1, \dots, P_{2n}\}$ be the preimage of $\{\overline{P_1}, \dots, \overline{P_n}\}$

Choose one pt q_j in each component of

$\mathbb{R}P^2 - \bigcup_{i=1}^{n} H_i$



Each q_j determines a dual line in $\mathbb{R}P^2$, hence a great circle in S^2, call it C_j.

Thm: $M \simeq$ ~~$(S^2 - \{P_1, \ldots, P_{2n}\})$~~ homotopy equ. the space obtained from $S^2 - \{P_1, \ldots, P_{2n}\}$ by attaching disks to the C_j's

Exercise ① Find the Huisman model for $\#$

② Show the Huisman model has the correct fundamental grp.

Lecture 2 Suciu (8)

Braids & Fundamental Grps of Arrangements

1. Braid Grps B_n
2. Pure Braids P_n
3. Braid Monodromy
4. Fundamental Gps

1.1 Braids (Artin 1920's)

Braid grp B_n = braids on n strings
with $\cdot$ = concatenation from top to bottom

- identity = trivial braid
- inverse = mirror image

Ex: $B_1 = 1$, $B_2 = \mathbb{Z}$, $B_3 = \ldots$

1.2 Presentation:

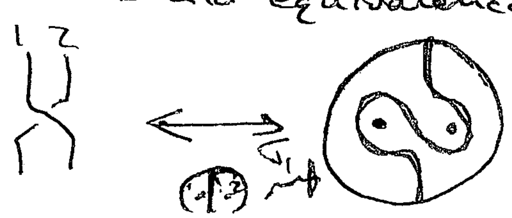
$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ $1 \le i \le n-2$ and

where $\sigma_i =$  $\sigma_i \sigma_j = \sigma_j \sigma_i$ $|i-j| \ge 2 \rangle$

1.3 Dynamical interpretation

braid $\leftrightarrow$ orientation preserving diffeomorphism of D^2, preserving pt. wise ∂D^2 and permuting n marked pts

braid equivalence $\leftrightarrow$ isotopy

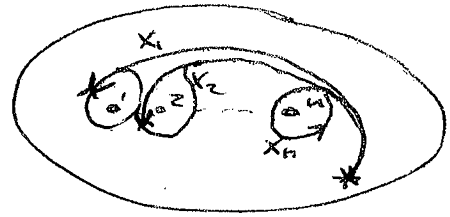
eg. 

upshot: $B_n = \mathcal{M}^n_{0,1}$
mapping class grps

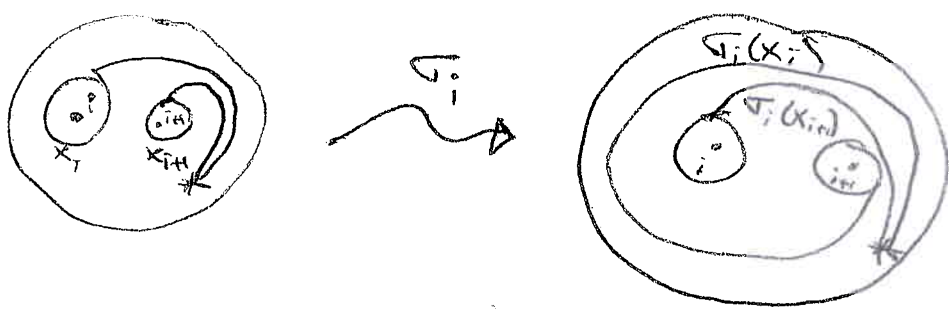
1.4 Artin Representation

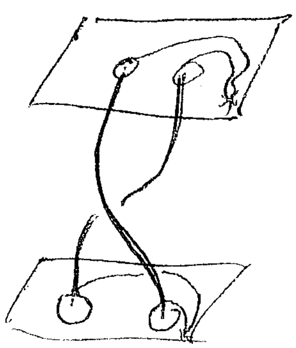
$$\alpha_n : B_n \hookrightarrow \mathrm{Aut}(F_n) \quad \text{injective homomorphism}$$

$$\beta \longmapsto \beta_* : \underset{\substack{\| \\ F_n}}{\pi_1(D^2 \setminus n\text{ pts})} \longrightarrow \underset{\substack{\| \\ F_n}}{\pi_1(D \setminus n\text{ pts})}$$



Compute $\quad \alpha_n(\sigma_i) = (\sigma_i)_*$





$$\sigma_i : \begin{cases} x_i \longmapsto x_i x_{i+1} x_i^{-1} \\ x_{i+1} \longmapsto x_i \end{cases}$$

$$\sigma_i^{-1} : \begin{cases} x_i \longmapsto x_{i+1} \\ x_{i+1} \longmapsto x_{i+1}^{-1} x_i x_{i+1} \end{cases}$$

boundary preserved

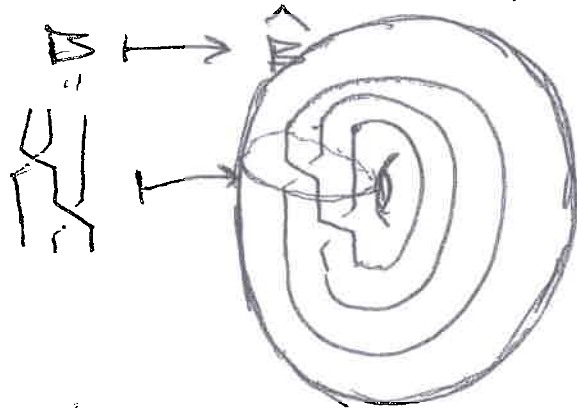
Remark: B_n is the subgp of $\mathrm{Aut}(F_n)$ consisting of $\varphi : F_n \longrightarrow F_n$ s.t.

- $\varphi(x_i) = w_i \, x_{\pi(i)} w_i^{-1}$
- $\varphi(x_1 \cdots x_n) = x_1 \cdots x_n$

1.5 Closed-up braids

Braids $\longrightarrow$ Links (of circles in S^3)

$B \longmapsto \hat{B}$



$D^2 \times S^1$

$\#$ components $= \#$ cycles in $\bar{B}$

Step 1: Trivial Bundle

$$D^2 \longrightarrow D^2 \times S^1$$
$$\downarrow$$
$$S^1$$

compute $\pi_1(S^3 \setminus \hat{B})$

Now remove $\hat{B}$ from $D^2 \times S^1$

$$D^2 \setminus \{n\,pts\} \longrightarrow D^2 \times S^1 \setminus \hat{B}$$
$$p \downarrow \uparrow s \quad \text{(section)}$$
$$S^1$$

(get non-trivial bundle with "monodromy" B i.e.

$$D^2 \times S^1 \setminus \hat{B} = (D^2 \setminus n\,pts) \times I \Big/ (x,0) \sim (B(x),1)$$

Review: $F \to E$, $E \downarrow B$

$$\cdots \to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \cdots$$

l.e.s. in π_*

$$\pi_2(S^1) \to \pi_1(D^2 \setminus \{n\,pts\}) \longrightarrow \pi_1(D^2 \times S^1 \setminus \hat{B}) \underset{s_*}{\overset{}{\rightleftarrows}} \pi_1(S^1) \longrightarrow 1$$

$\pi_2(S^1) = 1$, $\pi_1(D^2 \setminus \{n\,pts\}) = F_n$, $\pi_1(S^1) = \mathbb{Z}$

Since there is a section we have a splitting of the ses.

$$\pi_1(D^2\times S^1\setminus\hat{B}) = F_n \rtimes_B \mathbb{Z}$$

$$= \langle x_1,\dots,x_n,t \mid t^{-1}x_i t = B(x_i),\ i=1,\dots,n\rangle$$

Step 2: $S^3 = D^2\times S^1 \cup_{S^1\times S^1} S^1\times D^2$

So, $S^3\setminus\hat{B} = (D^2\times S^1\setminus\hat{B}) \cup_{S^1\times S^1} S^1\times D^2$

Now, use van Kampen

$$\begin{tikzcd}
 & \pi_1(S^1\times D^2) \arrow[dr] & \\
\pi_1(S^1\times S^1) \arrow[ur] \arrow[dr] & & \pi_1(S^3\setminus\hat{B}) \\
 & \pi_1(D^2\times S^1\setminus\hat{B}) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \overset{a}{\mathbb{Z}} \arrow[dr] & \\
\underset{a}{\mathbb{Z}}\times\underset{b}{\mathbb{Z}} \arrow[ur, "a \mapsto a,\ b \mapsto 1"] \arrow[dr] & & G \\
 & F_n\rtimes\mathbb{Z} \arrow[ur] &
\end{tikzcd}$$

$a \mapsto x_1\cdots x_n$

$b \mapsto t$

upshot

$$\pi_1(S^3\setminus\hat{B}) = \langle x_1,\dots,x_n \mid x_i = B(x_i),\ i=1,\dots,n\rangle$$

Example: (Hopf link)

Full twist on n strands $B_1 = \sigma_1^2$

$B_2 = (\sigma_1\sigma_2\sigma_1)^2$

$\hat{B}_n = \mathrm{Hopf}_n = L_{n,n}$

Compute $B(x_i) = x_i^{x_1 \cdots x_n}$

Hence $\pi_1(S^3 \setminus \hat{B} = L_{n,n}) = (x_1, \ldots, x_n \mid x_i = x_i^{x_1 \cdots x_n})$

$= (x_1, \ldots, x_n \mid x_1 \cdots x_n \text{ central})$

2. Pure Braid Grps

2.1 Generators for $P_n = \ker(B_n \to S_n)$

$A_{ij} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1} \quad 1 \le i < j \le n$

$A_{12} = \sigma_1^2 =$ 

$A_{13} = \sigma_1 \sigma_2^2 \sigma_1^{-1} =$ 

Relations: complicated

example: (exercise!) $\Delta_n^2 = (A_{12})(A_{13}A_{23})(A_{14}A_{24}A_{34}) \cdots (A_{1n}A_{2n} \cdots A_{n-1\,n})$

2.2 Topological interpretation

X - top. space

$F_n(X) = \{(x_1, \ldots, x_n) \in X \times \cdots \times X \mid x_i \ne x_j \ \forall i \ne j\}$

configuration space of n ordered pts in X.

S_n acts on $F_n(X)$ by permuting coordinates

$C_n(X) = F_n(X) / S_n$ = conf. space of n unordered pts in X

(13)

Have $S_n \to F_n(X)$

$\downarrow$ regular cover

$C_n(X)$

get

$(*) \quad 1 \to \pi_1(F_n(X)) \to \pi_1(C_n(X)) \to S_n \to 1$

Now take $X = \mathbb{C}$

$F_n(\mathbb{C})$ = complement of braid arr.

$\mathcal{A}_n = \{H_{ij}\}_{1 \le i < j \le n} \quad H_{ij} = \{z_i - z_j = 0\}$

$\downarrow$

$C_n(\mathbb{C})$ = discriminant space of monic polynomials of deg. n w/ no repeated roots

Notice that $(*)$ becomes

$$1 \to \underset{P_n}{\overset{\shortparallel}{\pi_1(F_n(\mathbb{C}))}} \to \underset{B_n}{\overset{\shortparallel}{\pi_1(C_n(\mathbb{C}))}} \to S_n \to 1 \qquad \text{(Fox)}$$

reason (braid B) $\longleftrightarrow$ ~~path~~ loop in conf. space $C_n(\mathbb{C})$

braid equivalence $\longleftrightarrow$ homotopy of loops

2.3 Semi direct product structure on P_n

$\mathbb{C} \to \mathbb{C}^n \xrightarrow{p_n} \mathbb{C}^{n-1}$ trivial bundle

$(z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-1})$

Restrict to conf. spaces

$\mathbb{C} \setminus (n-1 \text{ pts}) \to F_n(\mathbb{C}) \xrightarrow{p_n} F_{n-1}(\mathbb{C})$ (non-trivial) bundle w/ fiber $\mathbb{C} \setminus$ n-1 pts

$p_n^{-1}(z_1, \dots, z_{n-1}) = \{(z_1, \dots, z_n) \mid z_n \ne z_1, \dots, z_n \ne$

Moreover p_n admits a section

Use these bundles to understand $F_n(\mathbb{C})$ and

$P_n = \pi_1(F_n(\mathbb{C}))$ inductively

$F_1(\mathbb{C}) = \mathbb{C}$

$F_2(\mathbb{C}) = \mathbb{C} \times \mathbb{C}^*$

$\vdots$

$$
\begin{tikzcd}
F_n \arrow[r] & F_{n-1} \arrow[r] & \cdots \arrow[r] & F_2 \arrow[r] & F_1 = \mathbb{C} \\
\mathbb{C}_{n-1} \arrow[u] & \mathbb{C}_{n-2} \arrow[u] & & \mathbb{C}_1 \arrow[u] &
\end{tikzcd}
$$

get
- $F_n(\mathbb{C}) = K(P_n, 1)$
- $P_n = F_{n-1} \rtimes_{A_{n-1}} P_{n-1} = F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_2 \rtimes F_1$

e.g. $P_2 = F_1$

$P_3 = F_2 \rtimes F_1 \cong F_2 \times F_1$

$P_4 = F_3 \rtimes F_2 \rtimes F_1 \not\cong F_3 \times F_2 \times F_1$

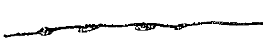
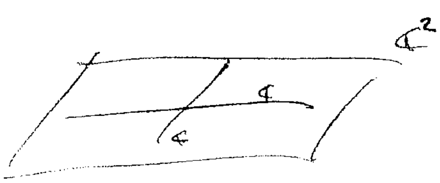
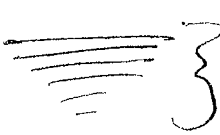
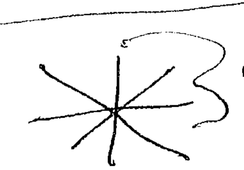
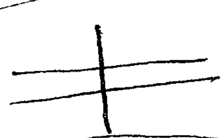
Lecture 3 — Suciu

Fundamental Grps of Arrangements

$\mathcal{A} = \{H_1, \ldots, H_n\}$ hyp. arr. in $\mathbb{C}^\ell$

$M = M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup H_i$, $G = G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$.
(path connected)

Examples

	$\mathcal{A}$	$M(\mathcal{A})$	$G(\mathcal{A})$
$\ell=1$	$\cdot$ $\mathbb{C}$ 0	$\simeq S^1$	$\mathbb{Z}$
	 n pts	$\simeq \bigvee S^1$	F_n
2		$\mathbb{C}^* \times \mathbb{C} \simeq S^1$	$\mathbb{Z}$
	 n lines	$\mathbb{C} \setminus n$ pts	F_n
	 n lines	$S^2 \setminus (n \text{ pts}) \times S^1$	$F_{n-1} \times F_1$
		$\mathbb{C}^* \times (\mathbb{C} \setminus 2\text{pts})$	$F_2 \times F_1$
	 m	$\mathbb{C} \setminus (m \text{ pts}) \times \mathbb{C} \setminus (n \text{ pts})$	$F_m \times F_n$

ℓ $\mathcal{A}_\ell = \{H_{ij}\}_{1 \le i < j \le \ell}$ $F_\ell(\mathbb{C})$ $P_\ell = F_{\ell-1} \rtimes_{\alpha_{\ell-1}} F_{\ell-2} \rtimes \cdots \rtimes_{\alpha_2} F_1$

$H_{ij} = \ker\{z_i - z_j\}$ conf. space of

braid arr.

$n = \binom{\ell}{2}$ $\alpha_\ell : P_\ell \to \mathrm{Aut}(F_\ell)$

By taking a generic 2-slice, we can assume $\ell = 2$.

$$\mathcal{A} \text{ in } \mathbb{C}^\ell \xrightarrow{\text{slice}} \mathcal{A}' \text{ in } \mathbb{C}^2$$

set of hyperplanes lines

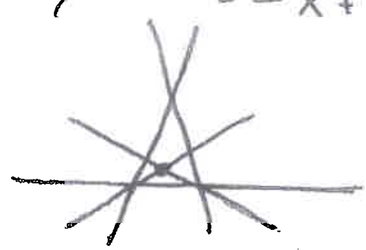
By a theorem of Zariski & Lefschetz, as proved by Hamm & Lê

$$\pi_1(M(\mathcal{A})) = \pi_1(M(\mathcal{A}'))$$

e.g. $\mathcal{A}_4$ braid arr. in $\mathbb{C}^4$

$\mathcal{A}_4 = \mathcal{A} \times \mathbb{C}$ where $\mathcal{A}$ is given by $Q(\mathcal{A}) = xyz(x-y)(x-z)(y-$

take a generic section say $z = x + 3y + 59$ get



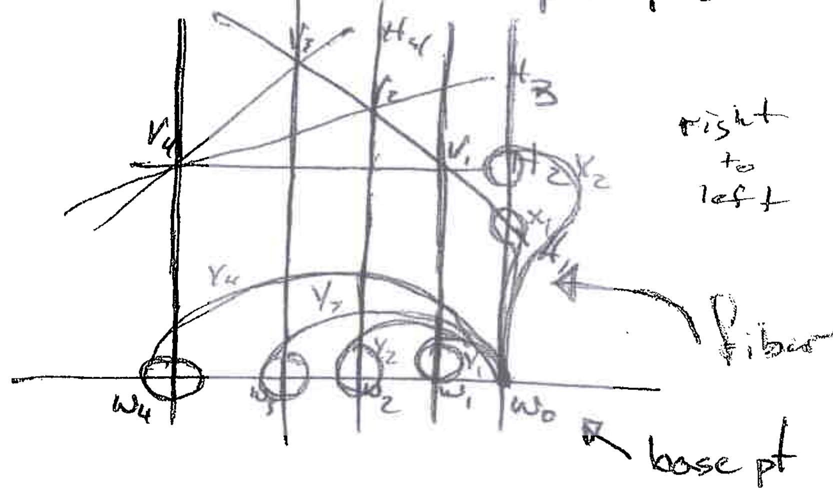
So, now take $\mathcal{A} = \{H_1, \dots, H_n\}$ affine arr. of lines in $\mathbb{C}^2$. By making a change of coord. we may assume

$p : \mathbb{C}^2 \longrightarrow \mathbb{C}$

$p(x,y) = x$ is generic w.r.t. $\mathcal{A}$.

$\Longleftrightarrow$ $p^{-1}(x)$ does not contain a line H_i and not more than one multiple pt.

$$\begin{array}{c} \mathbb{C}^2 = (x,y) \\ \Big\downarrow p \\ \mathbb{C} = (x) \end{array}$$



$V_1, \ldots, V_s$ multiple pts of $\mathcal{A}$

$w_k = p(V_k)$

$L_k = p^{-1}(w_k)$

$\hat{\mathcal{A}} = \{H_1, \ldots, H_n, L_1, \ldots, L_s\}$

$M(\hat{\mathcal{A}})$ complement

$$\begin{array}{ccc} \mathbb{C} \setminus (n \text{ pts}) & \longrightarrow & M(\hat{\mathcal{A}}) \\ & & \Big\downarrow \\ & & \mathbb{C} \setminus (s \text{ pts}) \end{array}$$

is a bundle

To show this is a bundle we use the diagram

$$\begin{tikzcd}
\text{fiber} & \mathbb{C}\setminus(n\ \text{pts}) \arrow[rr, equal] \arrow[d] & & \mathbb{C}\setminus(n\ \text{pts}) \arrow[d] \\
& \mathcal{M}(\mathcal{A}_{n+1}) = F_{n+1}(\mathbb{C}) \arrow[d] & & \mathcal{M}(\hat{\mathcal{A}}) \arrow[ll] \arrow[d] \\
\text{section} & \mathcal{M}(\mathcal{A}_n) = F_n(\mathbb{C}) \arrow[u, bend left, "s_{n+1}"] & & \mathbb{C}\setminus(s\ \text{pts}) \arrow[ll, "b"'] \arrow[u, bend right, "\sigma"']
\end{tikzcd}$$

where $b = (b_1, \ldots, b_n)$ where

$Q(\hat{\mathcal{A}}) = (y - b_1(x)) \cdots (y - b_n(x))$ $\qquad b_i(x) = m_i x + c_i$

Complement is total space of the pullback bundle

$$b_* : \pi_1(\mathbb{C}\setminus(s\ \text{pts})) \longrightarrow \pi_1(\mathcal{M}(\mathcal{A}_n))$$

the induced homom

$$B \;:\; F_s \longrightarrow P_n \qquad B = (B_1, \ldots, B_s)$$

$B_k \in P_n$

is called the <u>braid</u> <u>monodromy</u> of $\hat{\mathcal{A}}$.

L.e.s. in π_* for the bundle

$$1 \to \underset{F_n}{\pi_1(\mathbb{C}\setminus(n\ \text{pts}))} \to \pi_1(\mathcal{M}(\hat{\mathcal{A}})) \underset{\sigma_*}{\rightleftarrows} \underset{F_s}{\pi_1(\mathbb{C}\setminus(s\ \text{pts}))} \to 1$$

thus $\pi_1(\mathcal{M}(\hat{\mathcal{A}})) = F_n \rtimes_B F_s$

where $B : F_s \to P_n \overset{\alpha_n}{\hookrightarrow} \mathrm{Aut}(F_n)$ $\qquad = \left(x_1, \ldots, x_n, y_1, \ldots, y_s \mid y_k^{-1} x_i y_k = B_k(x_i) \right)$

Filling in $\bigcup L_k \setminus V_k$ gives back $M(\mathcal{A})$

(i.e. $M(\mathcal{A}) = M(\tilde{\mathcal{A}}) \cup \bigcup_{k=1}^{s} (L_k - V_k)$)

Use van Kampen to get

$$\boxed{\pi_1(M(\mathcal{A})) = (x_1, \dots, x_n \mid x_i = \beta_k(x_i))}$$

$k = 1, \dots, s$

$i = i_1, \dots, i_{m_k - 1}$

where m_k = multiplicity of V_k

if $V_k = H_{i_1} \cap \dots \cap H_{i_{m_k}}$

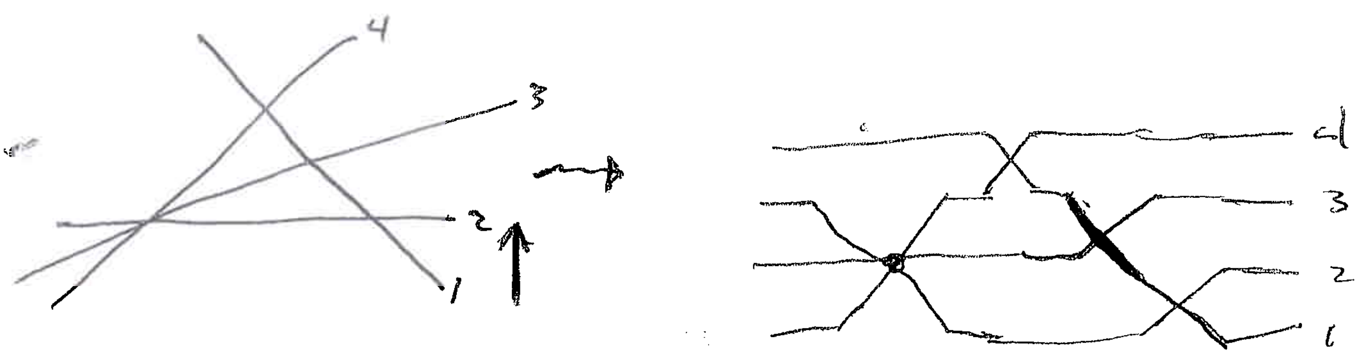
This is the minimal presentation for π_1. It carries the homotopy type of $M(\mathcal{A})$.

$$M(\mathcal{A}) \simeq \text{presentation 2-complex}$$

Explicit way to find braid monodromy generators

Read them off a "wiring diagram" (in complexified real case)

"braiding wiring diagram" (in general)



$\beta_4 = A_{234}$ $\quad \beta_3 = A_{14}$ $\quad \beta_2 = A_{13}$ $\quad \beta_1 = A_{12}$

where

where if $I = (i_1, \ldots, i_r)$ (20)

$$A_I = A_{i_1 i_2} A_{i_1 i_3} A_{i_2 i_3} \cdots A_{i_1 i_r} \cdots A_{i_{r-1} i_r}$$

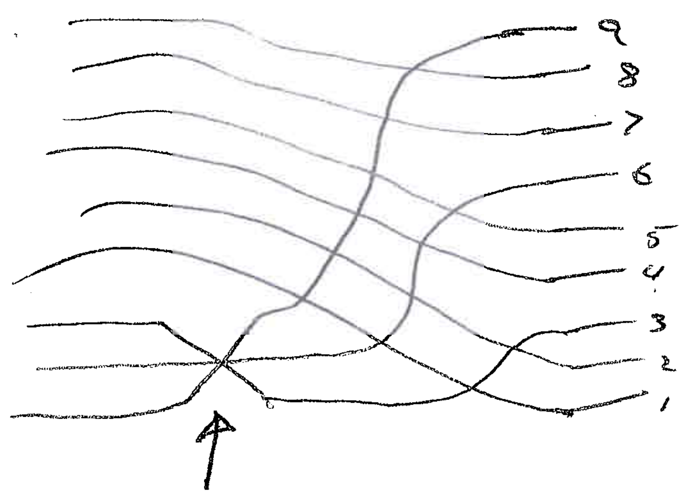
in general:

$$B_K = A_{I_K}^{\delta_K} \qquad \left(\begin{array}{c}\text{notation}\\ a^b = b^{-1}ab\end{array}\right)$$

where $I_K = \{$ indices of wires meeting at $V_K \}$

$$\delta_K = \prod_{i \in I_K} \prod_{j \in J_K} A_{ji}$$

where J_K = indices of middle wires that go above I_K at V_K and intersect the intersect wires



$$B = A_{369}^{A_{46} A_{56} A_{49} A_{59} A_{79} A_{89}}$$

$\boxed{xy = -yx}$

$k\langle e_1, \dots, e_n\rangle / e_ie_j + e_je_i$

(1) $\mathcal{A} = \{H_1, \dots, H_n\}$, $H_j \overset{\text{codim } 1}{\subset} \mathbb{C}^\ell$

Fix K = a comm. ring

Let E be the exterior alg. over K with generators $e_1, \dots, e_n$; $\deg e_i = 1$

$E = \bigoplus_{i=0}^{n} E_i$ (as free K-mod) $\deg E_i = i$

$E_0 = K$, $E_1 = \bigoplus_{j=1}^{n} Ke_j$, $E_p = \overset{p}{\Lambda} E_1$, $e_ie_j = -e_je_i$

$e_i^2 = 0$

$E_p = \bigoplus_{|S|=p} K \underbrace{e_{i_1} \cdots e_{i_p}}_{e_S}$ $i_1 < \cdots < i_p$ $S = \{i_1, \dots, i_p\}$

Define $d: E \to E$, K-linear, $d^2 = 0$, $\deg d = -1$

$d: E_p \to E_{p-1}$ $d(ab) = d(a)b + (-1)^{\deg a} a\,d(b)$

if a is homogeneous.

$de_i = 1$, $\forall i$.

It implies $de_S = \sum (-1)^{j-1} e_{S_j}$ $S_j = S \setminus \{i_j\}$

$I(\mathcal{A}) = (de_S \mid S = \underbrace{\text{minimal dependent set}}_{\text{circuits}})$

ideal of E

Ex: $\overset{1}{x}, \overset{2}{y}, \overset{3}{z}, \overset{4}{x-y}, \overset{5}{x-z}, \overset{6}{y-z}$

$e_1, \dots, e_6$

Dep. sets: 124, 135, 236, 456, 1256, ...

get as a generator of I

$e_1e_2 - e_1e_4 + e_2e_4 = (e_1-e_2)(e_1-e_4)$

$e_1e_3 - e_1e_5 + e_3e_5$

$e_2e_3 - e_2e_6 + e_3e_6$

$e_4e_5 - e_4e_6 + e_5e_6$

Pr 1: Show that these 4 generate $I(\mathcal{A})$.

$$\boxed{A = A(\mathcal{A}) = E/I(\mathcal{A})}$$

O.S.-alg. (Orlik-Solomon)

P2: If S is dep. then $e_S \in I$

② Dependence of Combinatorics

$A(\mathcal{A})$ is defined by the matroid of $\mathcal{A}$.

= intersection lattice of $\mathcal{A}$ (= the set of all $\cap$'s of H_i's ordered opposite to inclusion).

Dimensions:

	0	1	2	3	4	5	6
E	1	6	15	20	15	6	1
A	1	6	11	6	0	0	0

$$H(\mathcal{A},t) = \sum_{p=0}^{\ell} \dim(A_p)t^p = 1 + 6t + 11t^2 + 6t^3$$

$$= (1+t)(1+2t)(1+3t)$$

The lattice

$\operatorname{rk} X = \operatorname{codim} X$

$X \in L$

A set S of atoms is ~~dependent~~ iff

~~dep~~ $\operatorname{rk} \vee(S) < |S|$

a) A is graded by L, $\forall x \in L$

$$A_x = \langle e_S \mid \vee S = x \rangle_K$$

Pr 3 Prove A_x is well-defined.

$$A = \bigoplus_{\substack{k\text{-mod} \\ x \in L}} A_x \; ; \quad A_p = \bigoplus_{\operatorname{rk} x = p} A_x$$

b) A is filtered by L. $A(X) = A(\mathcal{A}_x)$

where $\mathcal{A}_x = \{ H \in \mathcal{A} \mid x \leq H \}$ $\quad A \supset A(x)$

Pr 4 Prove $A \supset A(x)$.

Filtration is defined by the grading

$$A(x) = \bigoplus_{y \leq x} A_y$$

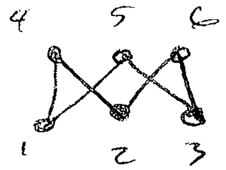
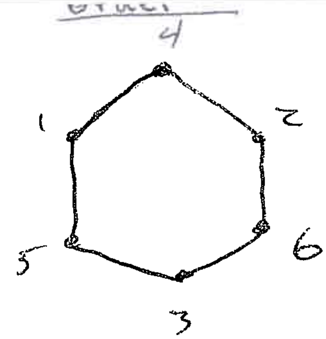
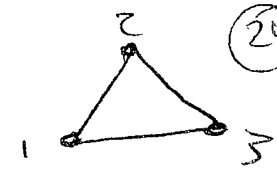
c) Homological Interpretation.

P = poset (finite)

The ~~flag~~ (order) complex:

simplices ~~are~~ flags in P

= linearly ordered subset

Ex P   

Via this, P gets alg. top.!

There are other complexes for P homotopy eq. to this.

Eg. If P is a lattice it is atomic complex (with highest and lowest pts deleted)

= simplices are sets of atoms bounded from above but not by highest element.

PrS: Prove homotopy equivalence, of flag and atomic complexes for a lattice.

We need the relative atomic complex

~~(Δ simplex on atoms, atomic complex) or~~ the chain complex D

where $D_p = \langle \sigma \subseteq A, |\sigma| = p \rangle_K$

and

$$d : D_p \longrightarrow D_{p-1}$$

$$d(\sigma) = \sum (-1)^{j-1} \sigma_j \qquad \sigma_j = \sigma \setminus \{H_{i_j}\} \quad \sigma = \{H_{i_1}, \ldots, H_{i_p}\}$$

$d^2 = 0$ $\qquad$ $V\sigma_j = V\sigma$

① $\underset{\substack{\text{as}\\ \text{chain}\\ \text{complex}}}{D} = \bigoplus_{x \in L} D_x$

where $D_x = \langle \sigma \mid V(\sigma) = x \rangle$

② $D_x = \left(\Delta_x , \begin{smallmatrix}\text{atomic}\\ \text{complex}\end{smallmatrix} \right)$

more precisely $\forall x$ we have exact:

$$0 \to D'(x) \to D''(x) \longrightarrow \underset{\uparrow \text{ sub. } x}{D(x)} \longrightarrow 0$$

$D'(x) = \begin{smallmatrix}\text{atomic}\\ \text{comp.}\\ \text{on } A_x\end{smallmatrix}$ $\qquad D''(x) = \text{simplex on } A_x$

$$\Rightarrow \boxed{H_p(D_x) = \tilde{H}_{p-2}(L_x)} \qquad L_x = \{ y \in L \mid y \leq x \}$$

③ <u>Folkmann theorem</u> (1966)

For Γ-lattice

$$H_p(D_x) = \begin{cases} 0 & \operatorname{codim} x \neq p \\ K^{m_x} & \operatorname{codim} x = p \end{cases}$$

where $m_x = \mu_L(x)$

0 In the beginning

Def 0.1 V an ℓ-dim v.s. $/K$

$\mathcal{A}$ is a (central) arrangement of hyperplanes

$\Updownarrow$

$\mathcal{A}$ is a finite collection of one-codimensional vector subspaces of V.

Def 0.2 $L(\mathcal{A}) := \{\bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}\}$ Agree $\bigcap_{H \in \emptyset} H = V$

$\uparrow$ poset $\qquad X \leq Y \overset{\text{def}}{\Longleftrightarrow} X \supseteq Y$

V the minimum in $L(\mathcal{A})$

$T(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} H$ is the maximum of $L(\mathcal{A})$

Def 0.3 $\mu : L(\mathcal{A}) \to \mathbb{Z}$ defined inductively by

$\mu(V) = 1$

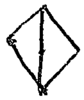
$$\mu(X) = -\sum_{\substack{Y \leq X \\ Y \neq X}} \mu(Y) \qquad (X \neq V)$$

Pr 1: $\mathcal{A} = \mathcal{A}_4$ Find $T(\mathcal{A})$ and $\mu(T(\mathcal{A})) = 6$

Def 0.4

$$\underline{\mathrm{Poin}(\mathcal{A}, t)} = \sum_{X \in L} |\mu(X)| t^{\operatorname{codim} X}$$

$$= \sum_{X \in L} (-1)^{\operatorname{rk} X} \mu(X) t^{\operatorname{rk} X}$$

ex:  $\pi(\mathcal{A}, t) = 1 + 3t + 2t^2 = (1+t)(1+2t)$

1. Finite Reflection Grps

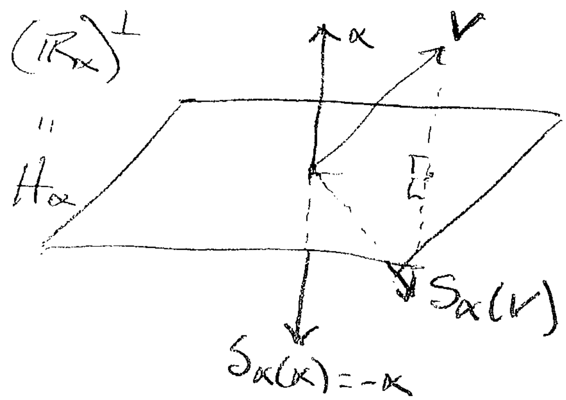
V ℓ-dim Euclidean space with inner product $(\,,\,)$ $/\mathbb{R}$

Let $\alpha \neq 0 \in V$

$$S_\alpha \in O(V) \quad \begin{cases} S_\alpha(\beta) = \beta \text{ if } (\alpha,\beta)=0 \\ S_\alpha(\alpha) = -\alpha \end{cases}$$

Then

$$\boxed{S_\alpha(v) = v - 2\frac{(\alpha,v)}{(\alpha,\alpha)}\alpha}$$



S_α: (orthogonal) reflection

$S_\alpha^2 = Id$

Def 1.1 Notation $S_\alpha = S_{H_\alpha} = S_H$

$H = \ker(I - S_H) = Fix(S_H)$

Def 1.2 $W \subset O(V)$

W is finite reflection grp $\Leftrightarrow$ W is a finite subgrp of $O(V)$ generated by reflections

2. Reflection Arrangements

Def 2.1 $\mathcal{A}$ is an arr. of hyp. in V

$\mathcal{A}$ is reflection arr. $\Leftrightarrow$ $S_H(\mathcal{A}) = \mathcal{A}$ $\forall H \in \mathcal{A}$

$$\left\{\begin{matrix}\text{finite}\\ \text{refl. grps}\end{matrix}\right\} \underset{\longleftarrow}{\xrightarrow{\text{one to one correspondence}}} \left\{\begin{matrix}\text{refl.}\\ \text{arrang.}\end{matrix}\right\}$$

$$W \longmapsto \mathcal{A}(W) = \{\ker(1-s) \mid s \in W \text{ is a reflection}\}$$

$$W(\mathcal{A}) \longleftarrow\!\!\!\shortmid \ \mathcal{A}$$

$$\| $$

$$\langle S_H \mid H \in \mathcal{A} \rangle$$

<u>Pr 2</u> Show $W(\mathcal{A})$ is a finite refl. grp if $\mathcal{A}$ is a refl. arr.

<u>Pr 3</u>: Which is the more obvious?

1. $\mathcal{A}(W(\mathcal{A})) = \mathcal{A}$ for any refl. arr.
2. $W(\mathcal{A}(W)) = W$ ------- grp

Prove the more obvious one and discuss the other.

<u>Examples</u> $(\mathcal{A}_\ell)$ braid arr.

$e_1, \dots, e_{\ell+1}$ orthogonal basis for $\mathbb{R}^{\ell+1}$

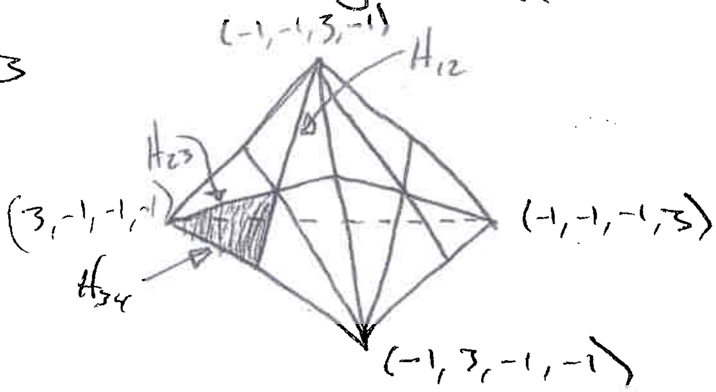
$x_1, \dots, x_{\ell+1}$ the dual " -- $(\mathbb{R}^{\ell+1})^*$

$V = \ker(x_1 + \dots + x_{\ell+1}) = \{c_1 e_1 + \dots + c_{\ell+1} e_{\ell+1} \in V \mid c_1 + \dots + c_{\ell+1} = 0\}$

$\cong \mathbb{R}^\ell$

$\mathcal{A}_\ell := \{H_{ij} \mid i < j\}$ $\quad H_{ij} = \ker(x_i - x_j) \cap V$

$\mathcal{A}_3$



24 chambers

6 planes

<u>Pr 4</u>: Show $\mathcal{A}_\ell$ is a refl. arr. Find $|\mathcal{A}_\ell|$ and $|W(\mathcal{A})|$ and what grp is $W(\mathcal{A})$?

Example (B_ℓ) $\ell \geq 2$

$e_1, \dots, e_\ell$ O.N. basis of $\mathbb{R}^\ell = V$

$x_1, \dots, x_\ell$ dual basis

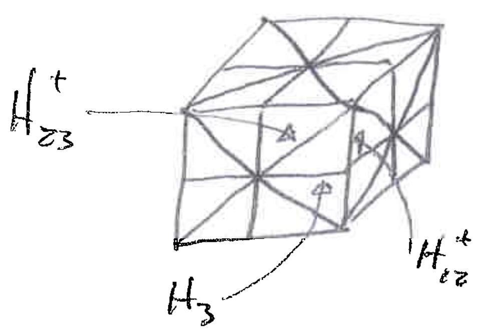
$H_{ij}^+ = \ker(x_i - x_j) \quad (i<j)$

$H_{ij}^- = \ker(x_i + x_j) \quad (i<j)$

$H_k = \ker(x_k)$

$$B_\ell = \{ H_{ij}^+, H_{ij}^-, H_k \mid 1 \leq i < j \leq \ell, 1 \leq k \leq \ell \}$$

Pr. 5 Same as Pr. 4 for B_ℓ



9 planes
48 chambers
$V = \mathbb{R}^3$

B_3

Example (D_ℓ) $\ell \geq 4$

Same notation as B_ℓ

$e_1, \dots, e_\ell$ $\quad H_{ij}^+, H_{ij}^-$

$$D_\ell = \{ H_{ij}^+, H_{ij}^- \mid 1 \leq i < j \leq \ell \}$$

Pr 6 Same as Pr 4 for D_ℓ

3. Chambers

$\mathcal{A}$ arr. in $\mathbb{R}^\ell = V$

<u>Def 3.1</u> $M(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H$

A connected component of $M(\mathcal{A})$ is called a chamber.

$\mathrm{Cham}(\mathcal{A}) = \{\text{chambers of } \mathcal{A}\}$

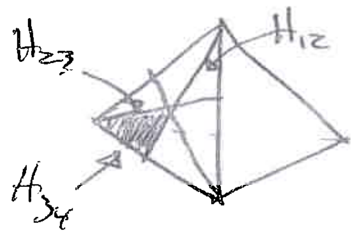
<u>Theorem 3.1</u> Let $C \in \mathrm{Cham}(\mathcal{A})$ and $\mathcal{A}$ is a refl. arr. and $T(\mathcal{A}) = \{0\}$

$\Rightarrow \exists!\ H_1, \dots, H_\ell \in \mathcal{A}$

$\exists!\ \alpha_1, \dots, \alpha_\ell \in V$ s.t.

- $H_i = (\mathbb{R}\alpha_i)^\perp \quad \forall i$
- $\|\alpha_i\| = 1 \quad \forall i$
- $C = \{v \in V \mid (v, \alpha_i) > 0 \quad i = 1, \dots, \ell\}$

In particular, each chamber is a simplicial cone



A_3

$H_1 = H_{12} \quad \alpha_1 = \frac{1}{\sqrt{2}}(e_1 - e_2)$

$H_2 = H_{23} \quad \alpha_2 = \frac{1}{\sqrt{2}}(e_2 - e_3)$

$H_3 = H_{34} \quad \alpha_3 = \frac{1}{\sqrt{2}}(e_3 - e_4)$

<u>system of symple roots</u>

S.S.R.

Coxeter diagram

$$\underset{\alpha}{\circ} \overset{4}{\text{———}} \underset{\beta}{\circ}$$

Draw

$\circ \overset{2}{\text{——}} \circ \Rightarrow \circ \quad \circ$

$\|\alpha\| = \|\beta\| = 1$

$\circ \overset{3}{\text{——}} \circ \Rightarrow \circ \text{——} \circ$

$(\alpha, \beta) = -\cos(\pi/4)$

$\circ \overset{n}{\text{——}} \circ \Rightarrow \circ \overset{n}{\text{——}} \circ$

$n \geq 4$

For the example A_3

$$\underset{\alpha_1}{\circ} \text{——} \underset{\alpha_2}{\circ} \text{——} \underset{\alpha_3}{\circ}$$

Pr 7: Find SSR and the Coxeter diagram for B_3

Theorem 3.2: $\mathcal{A}$ is refl. arr. and essential

$\alpha_1, \ldots, \alpha_\ell$ SSR

$\Rightarrow W = \langle s_{\alpha_1}, \ldots, s_{\alpha_\ell} \rangle$

Thm 3.3: W acts on $\mathrm{Cham}(\mathcal{A})$ transitively and effectively.

So, $\forall\, C_1, C_2 \in \mathrm{Cham}(\mathcal{A})$

$\exists!\, w \in W$ s.t. $C_1 = w(C_2)$

Cor. $|\mathrm{Cham}(\mathcal{A})| = |W(\mathcal{A})|$

4. Parabolic Subgrps

Recall $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$

Thm 4.1: $X \in L(\mathcal{A})$ $\quad W_X := \{w \in W \mid \mathrm{Fix}(w) \supseteq X\}$

$$\Longrightarrow W_X = W(\mathcal{A}_X) \text{ and } \mathcal{A}(W_X) = \mathcal{A}_X$$

$$\begin{array}{ccc}
\left\{\begin{matrix}\text{finite} \\ \text{refl. grps}\end{matrix}\right\} & & \left\{\begin{matrix}\text{refl.} \\ \text{arr.}\end{matrix}\right\} \\
W & \longmapsto & \mathcal{A}(W) \\
W(\mathcal{A}) & \longleftarrow\!\shortmid & \mathcal{A} \\
W_X & \longleftrightarrow & \mathcal{A}_X
\end{array}$$

In particular, W_X is a finite refl. grp and $\mathcal{A}_X$ is a refl. arr.

Thm 4.2 $L(\mathcal{A}) = \{\mathrm{Fix}(w) \mid w \in W\}$

Pr. 8 $X \in L(\mathcal{A})$

$$|\mu(X)| = |\{w \in W \mid \mathrm{Fix}(w) = X\}|$$

(Note: $\sum_{w \in W} \det w = 0$)

5. Classification

Def 5.1 $\mathcal{A}_1$ arr. in V_1

$\mathcal{A}_2$ ------ V_2

$\Rightarrow \mathcal{A}_1 \times \mathcal{A}_2 = \{H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2\}$

$|\mathcal{A}_1 \times \mathcal{A}_2| = |\mathcal{A}_1| + |\mathcal{A}_2|$

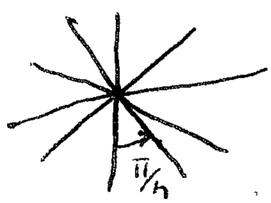
Def 5.2 $\mathcal{A}$ is called irreducible iff $\mathcal{A}$ can not be expressed as $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ with $\dim V_1 > 0$ & $\dim V_2 > 0$.

Thm 5.3: $\mathcal{A}$ is irr. refl. arr

$\Rightarrow \mathcal{A}$ is one of the following

$\ell = 1$ $\quad \mathcal{A}_1$

$\ell = 2$ $(n \geq 3)$ $\quad I_2(n)$



$A_2 = I_2(3)$

$B_2 = I_2(4)$

$G_2 = I_2(6)$

$\ell = 3$ $\quad \mathcal{A}_3, B_3, H_3$

$\ell = 4$ $\quad \mathcal{A}_4, B_4, D_4, F_4, H_4$

$\ell \geq 5$ $\quad A_\ell, B_\ell, D_\ell$

E_6

E_7

E_8

Pr 9: $V = \langle e_1, \dots, e_\ell \rangle$

$$[(e_i, e_j)] = \frac{1}{2} \underbrace{\begin{pmatrix} \frac{8}{9} & -\frac{1}{9} & \cdots & -\frac{1}{9} \\ -\frac{1}{9} & \frac{8}{9} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ -\frac{1}{9} & \cdots & & \frac{8}{9} \end{pmatrix}}_{\ell}$$

$$\mathcal{A}_\ell = \{\ker(x_i - x_j)\} \cup \{\ker(x_i + x_j + x_k)\}$$
$$\cup \{\ker(x_{i_1} + \dots + x_{i_6}) \mid 1 \le i_1 < \dots < i_6 \le \ell\}$$

$E_6 = \mathcal{A}_6$, $E_7 = \mathcal{A}_7$

$\underset{SSR}{\Delta} = \{e_i - e_{i+1} \mid i = 1, \dots, \ell-1\} \cup \{e_{\ell-2} + e_{\ell-1} + e_\ell\}$

Find $|E_6|$ & $|E_7|$. Show E_6 & E_7 are refl. arr. Find the Coxeter diagrams of E_6 & E_7.

6. Basic Invariants

$\mathcal{A}$ refl. arr. in V, $W = W(\mathcal{A})$, V^*-dual

Def: 6.1 $W \curvearrowright V^*$ contragradient action

$$\langle wx, v\rangle = \langle x, w^{-1}v\rangle \quad x \in V^*, v \in V, w \in W$$

Def 6.2 $S := S(V^*)$ the symmetric alg. of $V^*/\mathbb{R}$

Then $S = \mathbb{R}[x_1, \dots, x_\ell]$ is polynomials on V

$S = \bigoplus_{p=0}^{\infty} S_p \quad S_1 = V^* \quad W \curvearrowright S_p$

Thm 6.3 (Shephard-Todd 54, Chevalley 55)

$\mathcal{A}$ is (irr.) refl. arr.

$R = S^W$ the invariant subring

$\Rightarrow$ (1) $\exists\ \underbrace{f_1, \dots, f_\ell}_{\text{homog. alg. indep.}} \in R$ s.t. $R = \mathbb{R}[f_1, \dots, f_\ell]$

{basic invariants

(2) $\exists\ \underset{W\text{-stable}}{U} \subset S$ s.t. $S = U \otimes R$

(3) $W \curvearrowright U$ regular representation

$$\begin{array}{ccc} \mathbb{C}[x_1, \dots, x_\ell] & & \\ \cup & \xrightarrow{\text{Spec}} & \\ \mathbb{C}[x_1, \dots, x_\ell]^W = \mathbb{C}[f_1, \dots, f_\ell] & & \end{array}$$

$$\begin{array}{ccc} & \mathbb{C}^\ell & \\ \swarrow & & \searrow F = (f_1, \dots, f_\ell) \\ \mathbb{C}^\ell / W & \cong & \mathbb{C}^\ell \end{array}$$

the orbit space

$\underset{\substack{\text{the set of} \\ \text{critical pts}}}{\text{crit}(F)} = \bigcup_{H \in \mathcal{A}} H$

$\quad \| $

$\{ J(f_1, \dots, f_\ell) = 0 \}$

$J = \det\left[\dfrac{\partial f_i}{\partial x_j}\right]$

Vandermonde determinate

$$\mathbb{C}^\ell = \{ (x_1, \dots, x_{\ell+1}) \mid x_1 + \dots + x_{\ell+1} = 0 \}$$
$$\big\downarrow (f_1, \dots, f_\ell)$$
$$\mathbb{C}^\ell$$

$$f_j = \frac{1}{j+1} \sum_{k=1}^{\ell+1} x_k^{j+1}$$ power sum

$$\begin{vmatrix} x_1 & x_1^2 & \cdots & x_1^\ell \\ x_2 & & & \\ \vdots & & & \\ x_\ell & x_\ell^2 & \cdots & x_\ell^\ell \end{vmatrix} = \prod (x_i - x_j)$$

<u>Exponents</u> $f_1, \ldots, f_\ell$ basic invariants

$m_j = \underline{\deg} f_j - \underline{1}$ $\quad (j = 1, \ldots, \ell)$

$\mathcal{R}$ <u>irred.</u> refl. arr.

$1 = m_1 < m_2 \leq \cdots \leq m_{\ell-1} < m_\ell$

$h = m_\ell + 1$ is the <u>Coxeter #</u>

- $m_j + m_{\ell+1-j} = h \quad (j = 1, \ldots, \ell)$

$\mathrm{Poin}(R, t) = (1 + t^{m_1}) \cdots (1 + t^{m_\ell})$

Lecture 6 — Yuzvinsky

L, $D = D(L)$ $\quad D = \langle \sigma \subset \mathcal{A},\ \deg\sigma = |\sigma| \rangle_k$

$\mathcal{A}$ = atoms

$$d(\sigma) = \sum_{\substack{j=1 \\ \vee\sigma_j = \vee\sigma}}^{\deg\sigma} (-1)^{j-1}\sigma_j \qquad \tilde{H}_p(L) = H_{p+2}(D)$$

Folkman Thm $\quad L = \cap$-lattice

$$H_p(D_X) = \begin{cases} 0 & p \neq \operatorname{codim} X \\ |\mu(X)| & p = \operatorname{codim} X \end{cases}$$

P = poset

$$\mu = \delta^{-1},\ \mu_{X,Y} = \mu(X,Y) \qquad \delta = \begin{pmatrix} 1 & & 1 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$\mu(X) = \mu(V, X) \qquad \delta_{XY} = \begin{cases} 1 & X \leq Y \\ 0 & \text{otherwise} \end{cases}$$

Pr 1: Prove the 1st part of F.T.
(Hint: compute $\operatorname{codim} X - 2$ - skeleton of atomic complex of L_X)

Convert D to a diff. alg.

$$\sigma \cdot \tau = \begin{cases} 0 & \text{if } \operatorname{codim} \vee \sigma\cup\tau \neq \operatorname{codim} \vee\sigma + \operatorname{codim}\vee\tau \\ (\sigma\cup\tau)\,\varepsilon(\sigma,\tau) & \text{otherwise} \end{cases}$$

where $\varepsilon(\sigma,\tau)$ = sign of the shuffle of σ and τ

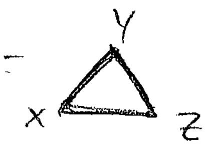
Pr 2: Check that $\bullet$ generates a DGA.

Note: $\forall i$ $H_i \in \mathcal{A}$ is a 1-cycle
$\quad \overset{\shortparallel}{\xi_i}$

Thm: The assignment $e_i \mapsto \xi_i$ generates a graded K-algebra iso. $A \xrightarrow{\sim} H_*(D)$

Cor. $$H(A,t) = \sum_{X\in L} |\mu(X)|\, t^{\operatorname{codim} X} = \operatorname{Poin}(L,t)$$

Ex: $\overset{1}{x}$ $\overset{2}{y}$ $\overset{3}{z}$ $\overset{4}{x-y}$ $\overset{5}{x-z}$ $\overset{6}{y-z}$

$K =$ 

$H(A,t) = 1 + 6t + 11t^2 + 6t^3$

$S.R.(K) = k[x,y,z]/(xyz)$

x^2, y^2, z

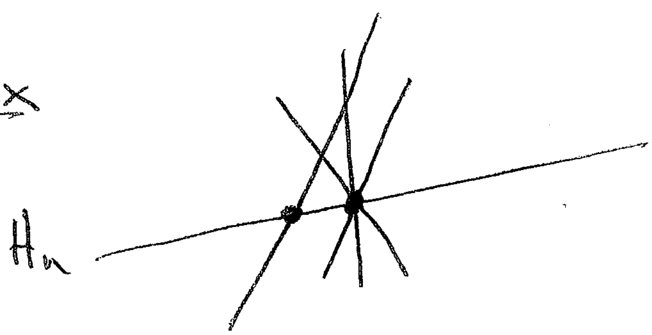
Deletion-Restriction Exact Seq.

$\mathcal{A}$, $\mathcal{A}' = \mathcal{A}\setminus\{H_n\}$, $\mathcal{A}'' = \{H_n \cap H \mid H \in \mathcal{A}\}$

$H_n \in \mathcal{A}$

Have OS.'s A, A', A''

Ex



There is the exact seq. of graded K-modules

$$0 \to A' \xrightarrow{i} A \xrightarrow{j} A''(-1) \to 0$$

Pr 3: i is an embedding of algebras.

$$j(e_S) = \begin{cases} e_{\lambda(S\setminus\{n\})} & \text{if } n\in S \\ 0 & \text{if } n \notin S \end{cases}$$

where $\lambda(H_i) = H_i \cap H_n$, $i \neq n$

(Pr 4)* Prove exactness

Topological interpretation

Thm: "$A(\mathcal{A}) \simeq H^*(M(\mathcal{A}))$"
as graded k-alg.
(Arnold, Brieskorn, O-S)

Have exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^*(M') \arrow[r] & H^*(M) \arrow[r] & H^{*-1}(M'') \arrow[r] & 0 \end{tikzcd}$$

$M \subset M'$, $M'' \subset M'$ $\quad M' \setminus M'' = M$

N = Tubular neighborhood of M'' in M'

$$\begin{tikzcd} N \supset \mathbb{C} \arrow[d, "\text{trivial}"] \\ M'' \end{tikzcd}$$

So, $N = M'' \times \mathbb{C}$, $N_0 = N \setminus M''$

So, $(N, N_0) = M'' \times (\mathbb{C}, \mathbb{C}^*)$

and $H^*(N, N_0) = H^*(M'') \otimes \underset{\substack{\wr\wr \\ H^1(\mathbb{C}^*) \simeq \mathbb{Z} \;=: \langle t \rangle}}{H^2(\mathbb{C}, \mathbb{C}^*)}$

$$H^*(N, N_0) \underset{+2}{\overset{\sim}{\leftarrow}} H^*(M'')$$

Exact seq. of (M', M)

$$\cdots \to H^p(M') \xrightarrow{i^*} H^p(M) \xrightarrow{\delta} H^{p+1}(M', N) \xrightarrow{j^*} H^{p+1}(M') \to \cdots$$

Excise $M' \setminus N$ from (M', M)

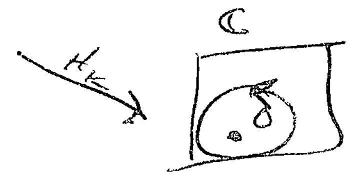
iso. $\alpha : H^*(M', M) \xrightarrow{\sim} H^*(N, N_0)$

$$\cong$$

$$H^*(M'')$$

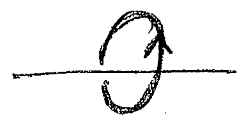
diff. form

$\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z}$



Pull back $\frac{dz}{z}$

$$\boxed{\omega_K = \frac{d\alpha_K}{\alpha_K}} \longleftrightarrow$$



$[\omega_K] \in H^1(M)$

Pr 5: Prove that $\left(\frac{d\alpha_K}{\alpha_K}\right)$ satisfy all the relations on e_i in A

This gives us the maps

$$e_i \longmapsto \omega_i$$

$$A \longrightarrow H^*(M).$$

Thm: Assignment $e_i \longmapsto \omega_i$ generates a graded alg. iso

$$A \xrightarrow{\sim} H^*(M)$$

Cor.: $P(M,t) = H(A,t) = \mathrm{Poin}(L,t)$

$$= \sum_{X \in L} |\mu(X)| t^{\operatorname{codim} X}$$

Lecture 7 Terao

Free Arrangements

V is ℓ-dim v.s. over $\mathbb{K}$

$\mathcal{A}$ a central arr. in V

$S = S(V^*)$, $Q = Q(\mathcal{A})$

1. Free Arr.

1. Def. 1.1 $\mathrm{Der}_S := \{\theta : S \to S \mid \theta \text{ is } \mathbb{K}\text{-linear derivati[on]}\}$

Put $\partial_i(x_j) = \delta_{ij}$ $\quad \mathrm{Der}_S \overset{\psi}{\cong} S^\ell$

So, $\forall \theta \in \mathrm{Der}_S$, $\theta = f_1\partial_1 + \cdots + f_\ell \partial_\ell$

Def 1.2 $D(\mathcal{A}) := \{\theta \in \mathrm{Der}_S \mid \theta(Q) \in QS\}$

Pr 1: Prove $D(\mathcal{A}) = \{\theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in \alpha_H S \ \forall H \in \mathcal{A}\}$

Geometric interpretation

$K = \mathbb{R}$ or $\mathbb{C}$

$\mathrm{Der}_S \longleftrightarrow$ polynomial vector fields on K^ℓ

$D(\mathcal{A}) \longleftrightarrow$ " " " on K^ℓ tangent to each hyperplane

$K = S_{(0)}$ the quotient field of S

Prop. $D(\mathcal{A}) \otimes_S K = \mathrm{Der}_S \otimes_S K \simeq K^\ell$

Pf: $Q\,\mathrm{Der}_S \subset D(\mathcal{A}) \subset \mathrm{Der}_S$

Def 1.4 $\mathcal{A}$ free $\iff D(\mathcal{A})$ is a free S-mod. (of rank ℓ)

$\theta_1, \dots, \theta_\ell \in D(\mathcal{A})$ homog. basis

$m_j = \deg \theta_j$

$$\mathrm{Der}_S = \bigoplus_{p=0}^{\infty} (\mathrm{Der}_S)_p$$

$$(\mathrm{Der}_S)_p = S_p d_1 \oplus \cdots \oplus S_p d_\ell$$

$m_1, \dots, m_\ell$ <u>exponents</u> of $\mathcal{A}$ $\quad \exp(\mathcal{A}) = (m_1, \dots, m_\ell)$

Prop. 1.5 $\theta_1, \dots, \theta_\ell \in D(\mathcal{A})$

$$\Longrightarrow \det[\theta_i(x_j)] \in QS$$

Pf: $H \in \mathcal{A}$ $\quad \alpha_H = \sum_{i=1}^{\ell} c_i x_i \quad (\exists j \text{ s.t. } c_j = 1)$

Then

$$\det[\theta_j(x_i)] = \det \begin{bmatrix} \theta_1(x_1) & \cdots & \theta_\ell(x_1) \\ \vdots & & \vdots \\ \theta_1(\alpha_H) & \cdots & \theta_\ell(\alpha_H) \\ \vdots & & \vdots \\ \theta_1(x_\ell) & \cdots & \theta_\ell(x_\ell) \end{bmatrix} \in QS$$

divisible by α_H

▨

Thm 1.6 (Saito's Criterion)

$\theta_1, \dots, \theta_\ell \in D(\mathcal{A})$ are a basis

$$\Updownarrow$$

$$\det[\theta_j(x_i)] \doteq Q$$

Cor. 1.7 $\exp(\mathcal{A}) = (m_1, \ldots, m_\ell) \Rightarrow |\mathcal{A}| = m_1 + \cdots + m_\ell$

Thm 1.8 (Zaslavsky '75)

$\mathcal{A}$: any arr. / $\mathbb{R}$

$\mathrm{Poin}(\mathcal{A}, 1) = |\mathrm{Cham}(\mathcal{A})|$

Theorem 1.9 (Factorization Thm '81)

$\exp(\mathcal{A}) = (m_1, \ldots, m_\ell)$

$$\mathrm{Poin}(\mathcal{A}, t) = \prod_{i=1}^{\ell} (1 + m_i t)$$

Cor 1.10 If $\mathcal{A}$ is free then $|\mathrm{Cham}(\mathcal{A})| = \prod_{i=1}^{\ell} (1 + m_i)$

Pr 2: Show that $Q = xyz(x+y+z)$ is not free a

Pr 3: Show that every 2-arr. is free.

Pr 4: $\Theta_E = \sum_{i=1}^{\ell} x_i \partial/\partial x_i \in D(\mathcal{A})$

Show that Θ_E can be part of a basis if $\mathcal{A}$ is non-empty. $\left(D(\mathcal{A}) = S\Theta_E \oplus \mathrm{Ann}(Q) \right)$

So, $\mathcal{A}$ non-empty free $\Rightarrow 1 \in \exp(\mathcal{A})$

2. Reflection Arr.

$\mathcal{A}$ irr. refl. arr. in V, $W = W(\mathcal{A})$

$R = S^W = \mathbb{R}[f_1, \ldots, f_\ell]$ $f_1, \ldots, f_\ell$ basic invariants

Def 2.1 $f \in S$, f is an antiinvariant

$\iff \underset{w \in W}{w(f)} = \det(w) f$

Thm 2.2 $S^{-W} = \{\text{antiinvariant}\} = RQ$

Thm 2.3 $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (m_1, \dots, m_\ell)$ and irr. refl. arr.

$(m_j = \underline{\deg} f_j - 1)$ exponents of W

Pf: $f_1, \dots, f_\ell$ basic invariants

$$(\, , \,) : V^* \times V^* \longrightarrow \mathbb{R}$$

$$(\, , \,) : \Omega^1 \times \Omega^1 \rightarrow S$$

$$\theta_j(*) = (df_j, d*)$$

$$\deg \theta_j = \deg f_j - 1 \quad \text{if } df = \sum_{i=1}^{\ell} \frac{\partial f}{\partial x_i} dx_i$$

θ_j is W-invariant, $\theta_j(Q) \in S^{-W} = RQ$

$\Rightarrow \theta_j \in D(\mathcal{A})$

$$\det[\theta_j(x_i)] = \det J(f_1, \dots, f_\ell) \doteq Q$$

by Saito we are done.

Pr 5: Find a basis for $D(B_\ell)$ $Q = x_1 \cdots x_\ell \prod (x_i^2 - x_j^2)$

Pr 6: " " " " (D_ℓ) $\quad " = \prod (x_i^2 - x_j^2)$

Def 2.4 (unitary refl. grps) $V = \mathbb{C}^\ell$

$s \in U(\ell)$ is a $\underline{\text{unitary}}$ $\underline{\text{refl.}}$ $\iff \dim \mathrm{Fix}(s) = \ell - 1$

So, $s = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\zeta} \end{pmatrix}$ where $|\zeta| = 1$.

$G < U(\ell)$ a finite subgrp generated by unitary reflections <u>irreducible</u>

$S^G = \mathbb{C}[\underbrace{f_1, \ldots, f_\ell}_{\text{homog. and alg. indep.}}]$

again

$m_j = \deg f_j - 1$ are the exponents of G.

$\mathcal{A}(G) = \{ \mathrm{Fix}(g) \mid g \in G \text{ is a unitary refl.} \}$

is unitary refl. arr.

<u>Thm 2.5</u> $\mathcal{A}(G)$ is a free arr. with

$$\exp(\mathcal{A}) = (n_1, \ldots, n_\ell) \text{ are the coexp.}$$
$$\#$$
$$\exp(G) = (m_1, \ldots, m_\ell)$$

<u>Cor 2.6</u> $\mathrm{Poin}(\mathcal{A}(G), t) = \prod_{i=1}^{\ell} (1 + n_i t)$

3. <u>What are free arrangements?</u>

1. reflection arr.'s (real or complex) and their deformations
2. fiber type arr. (super-solvable)
3. arr.'s contracted by "addition-deletion thm" (inductively free arr.)

Addition - Deletion Thm

$H_0 \in \mathcal{A}$ $\qquad \mathcal{A}' = \mathcal{A} \setminus \{H_0\}, \ \mathcal{A}'' = \mathcal{A}^{H_0} = \{H_0 \cap H \mid H \in \mathcal{A}'\}$

- $\mathcal{A}', \mathcal{A}''$ free w/ $\exp(\mathcal{A}'') \subset \exp(\mathcal{A}') \Rightarrow \mathcal{A}$ free and
  (addition) $\qquad \exp(\mathcal{A}'') \subset \exp(\mathcal{A})$

- $\mathcal{A}, \mathcal{A}''$ free $\quad \exp(\mathcal{A}'') \subset \exp(\mathcal{A}) \Rightarrow \mathcal{A}'$ free and
  (deletion) $\qquad \exp(\mathcal{A}'') \subset \exp(\mathcal{A}')$

Pr 6 or 7 Show that $\mathcal{A}_\ell$ is inductively free with $\exp(\mathcal{A}_\ell) = (1, 2, \dots, \ell)$

Conjecture (open) (1981) Does the freeness (K-fixed) of $\mathcal{A}$ only depend upon $L(\mathcal{A})$?

Falk

L super solvable $\Longleftrightarrow \exists$ maximal chain of modular elements

semimodular : $rk(X \vee Y) + r(X \wedge Y) \le r(X) + r(Y)$

geometric lattice

$\uparrow X \wedge Y \qquad \uparrow \cap\{H \mid X + Y \subseteq H\}$

(X, Y) modular pair $\Longleftrightarrow$ equality holds

X is modular if (X, Y) is modular $\forall Y$

Thm: Suffices to check for only those Y for which $X \vee L = I_L$

Cor.: X modular $\Rightarrow (\forall Y, \ X \cap Y = 0 \Rightarrow X + Y = \mathbb{C}^\ell$

X modular $\Longleftrightarrow X + Y \in L \quad \forall Y$

Ducia

$A = H^*(M(\mathcal{A}))$

- $\mathcal{A}$ super solvable $\underset{\text{Shelton-Yuz}}{\Longrightarrow}$ A is Koszul $\mathbb{k}$ has linear over A

  $\mathcal{A}$ sup - - - $\Longleftrightarrow$ A has quadratic Gröb. basis

http://dean.serenevy.net

Combinatorics & Topology of the Complement: Topological Classification

$\mathcal{A}$: arr. in $\mathbb{C}^\ell$, $M = M(\mathcal{A}) = \mathbb{C}^\ell - \bigcup \mathcal{A}$

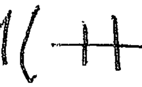
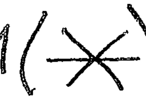
$L = \cap$ - lattice

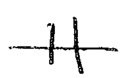
Thm: (i) $H^*(M) \cong A(\mathcal{A})$ as rings
(ii) $A(\mathcal{A})$ is uniquely determined by $L(\mathcal{A})$

Question 1: Is the homotopy type or homeomorphism type of M determined by $L(\mathcal{A})$?

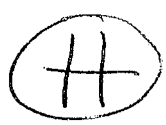
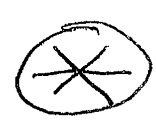
Answer: No

Ex: $\pi_1($  $) \cong F_2 \times F_1 \cong \pi_1($  $)$

In fact, $M($  $) \cong M($  $)$

$\mathcal{A}_1 =$  $\mathcal{A}_2 =$ 

$Q_1 = (x+1)(x-1)y$ $Q_2 = (x+y)(x-y)y$

"$c\mathcal{A}_1$:  $c\mathcal{A}_2$: 

$cQ_1 = (x+z)(x-z)yz$ $cQ_2 = (x+y)(x-y)yz$

$$M(\mathcal{A}_1) = M(d(c\mathcal{A}_1)) = \underbrace{M(c\mathcal{A}_1)/\mathbb{C}^*}_{\text{projective image}} \cong M(c\mathcal{A}_2)/\mathbb{C}^*$$

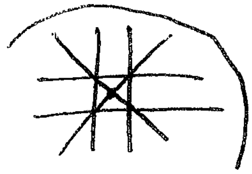
$$= M(d(c\mathcal{A}_2)) = M(\mathcal{A}_2)$$

Question 2: Are there central arr.'s for which $M(\mathcal{A}_1) \cong M(\mathcal{A}_2)$ but $L(\mathcal{A}_1) \not\cong L(\mathcal{A}_2)$?

OR: Are there central arr.'s for which the $A(\mathcal{A}_1) \cong A(\mathcal{A}_2)$ but $L(\mathcal{A}_1) \not\cong L(\mathcal{A}_2)$?

Answer: Yes. So, instead we ask the following: What combinatorial features of L can be extracted from A?

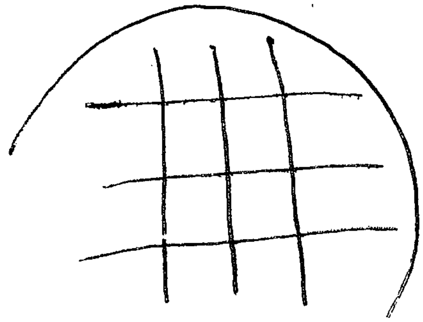
Observation 1 $A(\mathcal{A})$ determines $\mathrm{Poin}(L,t) = \sum_{X \in L} |\mu(X)| t^{\dots}$

Ex:  is inductively free $(1,3,3)$

$$\mathrm{Poin}(L,t) = (1+t)(1+3t)^2$$
$$= 1 + 7t + 15t^2 + 9t^3$$

The coeff.'s of $\mathrm{Poin}(L,t)$ are called "Whitney numbers of L of the 2^{nd} kind"

Ex:



supersolvable

$\mathrm{Poin}(L,t) = (1+t)(1+3t)^2$

The O.S. alg.'s are not iso. since one is supersolv. and the other isn't.

Construction: An invariant of the ring structure of $A(L)$.

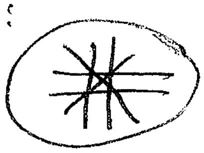
$A = E/I$, $A^1 \cong E^1$, $I^2 \cong \ker\left(\underset{\overset{\parallel}{E^2}}{\Lambda^2(A^1)} \to A^2\right)$

$\Delta : E^1 \otimes I^2 \longrightarrow E^3 = \Lambda^3(A^1)$

$x \otimes r \longmapsto x \wedge r$

Def: $\phi_3 = \dim(\ker \Delta)$

Ex:

 $\phi_3 = 17$

 $\phi_3 = 12$

$I^2 = \langle de_S \mid S \text{ dependent}, |S| = 3 \rangle$

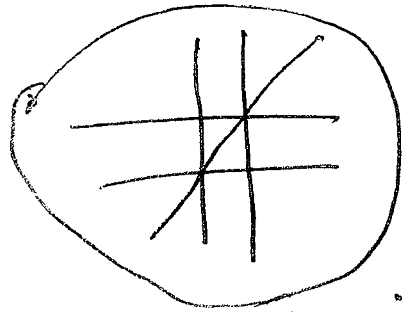
$= \langle de_{ijk} = (e_i - e_j) \wedge (e_i - e_k) \mid \{i,j,k\} \text{ dependent} \rangle$

$\Downarrow$

$(e_i - e_j) \otimes de_{ijk} \in \ker \Delta$

$(e_i - e_k) \otimes de_{ijk} \in \ker \Delta$

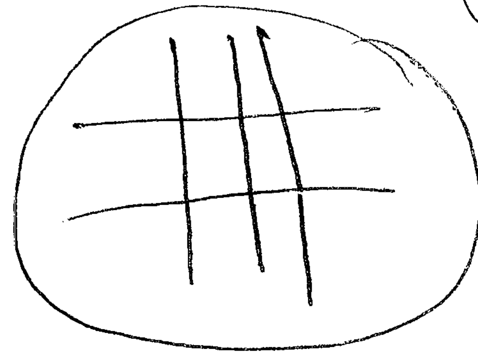
Ex



exp $= (1,2,3)$

both super-solvable

φ_3 is the same $=10$



exp $= (1,2,3)$

Thm (Sullivan, Morgan: rational htpy theory)

φ_3 = rank of the 3^{rd} factor in the lower central series of π_1

$$\pi = \pi^1, \ [\pi,\pi] = \pi^2, \ \dots, \ [\pi, \pi^k] = \pi^{k+1}$$

$$\varphi_3 = \mathrm{rank}\left(\pi^3 / \pi^4\right)$$

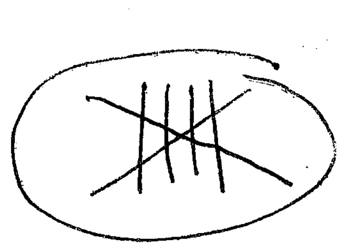
Thm (Falk, Randell) If $\mathcal{A}$ is supersolvable, then ranks of factors in the L.C.S. of π_1 are determined by the exponents.

Ex: $\pi_1(\#) = P_4$

$\pi_1(\#\!\#) = F_3 \times F_2 \times F_1$.

Question: Is $A(\mathcal{A}_1) \cong A(\mathcal{A}_2)$? Answer: No

Exercise: $\mathcal{A}$



show $\mathrm{Poin}(L(\mathcal{A}),t)$ splits over $\mathbb{Z}$ but $\mathcal{A}$ is not free.

<u>Question</u>: Are there arr.'s $\mathcal{A}_1$ and $\mathcal{A}_2$ for which $L(\mathcal{A}_1) \cong L(\mathcal{A}_2)$ but (i) $M(\mathcal{A}_1) \not\cong M(\mathcal{A}_2)$? Yes

(ii) $\pi_1(M(\mathcal{A}_1)) \not\cong \pi_1(M(\mathcal{A}_2))$? Yes

<u>Projective equivalence</u>

$$\mathcal{A} = \{H_1, \dots, H_n\} \text{ central} \qquad \alpha_i \in V^* \text{ s.t. } H_i = \ker(\alpha_i)$$

V^* w/ basis $x_1, \dots, x_\ell$

$$B = B(\mathcal{A}) = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}^{n \times \ell}$$

Then $\mathcal{A}_1$ is projectively equivalent to $\mathcal{A}_2$ iff $\exists$ $\ell \times \ell$ non-singular matrix C and $n \times n$ non-singular diagonal matrix D s.t.

$$B(\mathcal{A}_2) = D\, B(\mathcal{A}_1)\, C.$$

If $\mathcal{A}_1$ is projectively equi. to $\mathcal{A}_2$ then

$$M(\mathcal{A}_1) \cong M(\mathcal{A}_2)$$

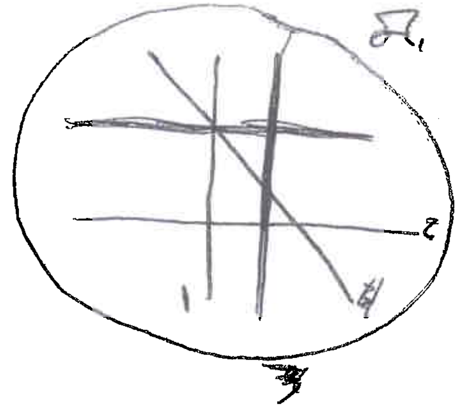
<u>Normal forms</u> (for central arr. of rank 3)

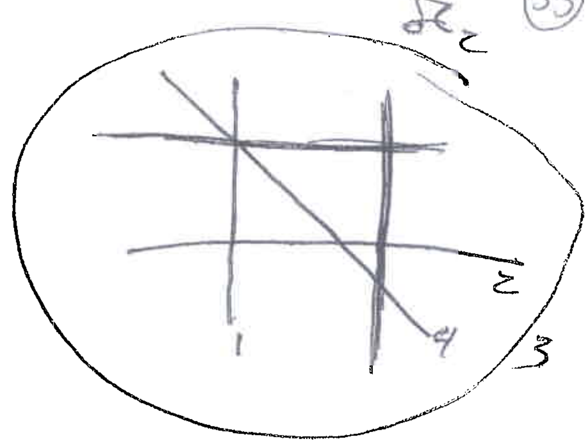
Assume H_1, H_2, H_3, H_4 are in "general position" (so it will look like ) (none $\{H_i, H_j, H_k\}$ are dependent)

Then $\mathcal{A}$ is projectively equ. to a unique arr. $\mathcal{A}_0$ with

$$B(\mathcal{A}_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Ex





$\mathcal{A}_1$ is not proj. eq. to $\mathcal{A}_2$

but $L(\mathcal{A}_1) \cong L(\mathcal{A}_2)$.

Is $M(\mathcal{A}_1) \cong M(\mathcal{A}_2)$?

In complex # we can slide 4 past the intersect to get isotopy of complime

Randell's Lattice-isotopy theorem

Suppose $\mathcal{A}_t = \{H_1(t), \ldots, H_n(t)\}$ $0 \le t \le 1$ is a family of arr.'s satisfying $\operatorname{codim}\left(\bigcap_{i \in S} H_i(t)\right)$ is constant for every S.

Then, $\exists$ a continuous change of variables carrying $(\mathbb{C}^l, \cup \mathcal{A}_0)$ to $(\mathbb{C}^l, \cup \mathcal{A}_1)$ and $M(\mathcal{A}_0)$ is diffeomorphi to $M(\mathcal{A}_1)$.

Fox Calculus & Alexander Invariants

G grp $\quad \mathbb{Z}G = \{ \sum_{g\in G} n_g g \mid n_g \in \mathbb{Z} \}$ finite

with multiplication given by the group

$\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ augmentation map
$g \mapsto 1$

$I = IG = \ker(\mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z})$ augm. ideal

$= \mathrm{span}(g-1 \mid g \in G\setminus\{1\})$

Assume G is finitely generated by $x_1, \dots, x_n$

Then $IG = \mathrm{span}(x_i - 1 \mid i=1,\dots,n)$

Is it a free $\mathbb{Z}G$-mod?

Answer: iff $G = F_n$ (Stallings)

We have $IF_n \cong (\mathbb{Z}F_n)^n$

top. hint $\tilde{C}_*(\overbrace{\vee^n S^1}) : 0 \to \overset{C_1}{\overset{\|}{(\mathbb{Z}F_n)^n}} \xrightarrow[\begin{pmatrix} x_1-1 \\ \vdots \\ x_n-1 \end{pmatrix}]{\partial_1} \overset{C_0}{\overset{\|}{\mathbb{Z}F_n}} \xrightarrow{\varepsilon} \mathbb{Z} \to 0$

snowflake $\qquad \underset{IF_n}{\|}$

Let $w \in F_n$

$$w - 1 = \sum_{i=1}^n \frac{\partial w}{\partial x_i}(x_i - 1)$$

coeff.s of $w-1$ in that basis expansion called Fox derivatives of w.

Extend linearly to

$$\partial/\partial x_i : \mathbb{Z}F_n \longrightarrow \mathbb{Z}F_n$$

uniquely characterized by two rules:

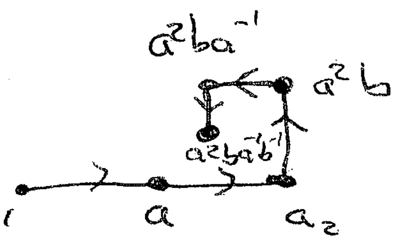
- $\dfrac{\partial x_j}{\partial x_i} = \delta_{ij}$

- $\dfrac{\partial (uv)}{\partial x_i} = \dfrac{\partial u}{\partial x_i}\,\varepsilon(v) + u \cdot \dfrac{\partial v}{\partial x_i}$

Additional properties

- $\dfrac{\partial 1}{\partial x_i} = 0$
- $\dfrac{\partial (x_i^{-1})}{\partial x_i} = -x^{-1}$

eg $\quad w = a^2 b a^{-1} b^{-1}$



$$\frac{\partial w}{\partial a} = 1 + a - a^2 b a^{-1}$$

$$\frac{\partial w}{\partial b} = a^2 - a^2 b a^{-1} b^{-1}$$

Resolution of $\mathbb{Z}$ over $\mathbb{Z}G$

Assume $G = (x_1, \dots, x_n \mid y_1, \dots, y_m) \overset{\varphi}{\twoheadleftarrow} F_n = (x_1, \dots$

$$\tilde{C}_*\left(\widetilde{K(G,1)}\right) \cdots \to (\mathbb{Z}G)^m \xrightarrow[\;J_G\;]{\partial_2} (\mathbb{Z}G)^n \xrightarrow[\;\begin{pmatrix} x_1 - 1 \\ \vdots \\ x_n - 1 \end{pmatrix}\;]{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where J_G is the Fox Jacobian of G

$$J_G = \left(\frac{\partial y_i}{\partial x_j}\right) \in \mathrm{Mat}_{n\times m}(\mathbb{Z}G)$$

Eg $G = (a,b \mid aba^{-1}b^{-1}=1) = \mathbb{Z}^2$

$K(G,1) = T^2$

$$\tilde{C}_*(\tilde{T}^2) \qquad \mathbb{Z}\mathbb{Z}^2 \xrightarrow{(1-b\ \ a-1)} (\mathbb{Z}\mathbb{Z}^2)^2 \xrightarrow{\binom{a-1}{b-1}} \mathbb{Z}\mathbb{Z}^2 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

Exercise (optional) Compute a free $\mathbb{Z}\mathbb{Z}^n$- resolution of $\mathbb{Z}$ (Koszul reso.)

Alexander Module/Invariant

X top space (path-conn.)

$G = \pi_1(X,*)$ $\qquad G' = [G,G]$

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G/G' \longrightarrow 1$$

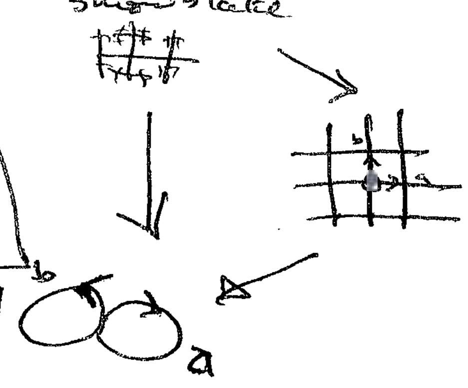
$$G/G' = G^{ab} = H_1(X)$$

eg. $X = M(\mathcal{A})$ $\quad H_1(X) = \mathbb{Z}^n$; $|\mathcal{A}| = n$

Let $G^{ab} \curvearrowright \tilde{X}$ be the universal abel. cover

$$\tilde{X} \xrightarrow{p} X$$

eg. snowflake



$A = H_1(\tilde{X}, p^{-1}(*)) = \mathbb{Z}G^{ab} \otimes_{\mathbb{Z}G} IG$ $\boxed{\text{Alex. mod.}}$

$B = H_1(\tilde{X}) = G'/G''$ $\boxed{\text{Alex. Invariant}}$

veiwed as modules over $\mathbb{Z}G^{ab}$

Exercise $0 \to B \to A \to I \to 0$ (Crowell exact seq.)

Now assume

$$\begin{tikzcd} G = (x_1, \dots, x_n \mid r_1, \dots, r_m) \arrow[d, "ab"] & x_i \arrow[d, mapsto] \\ G^{ab} = \mathbb{Z}^n = (t_1, \dots, t_n \mid [t_i, t_j] = 1) & t_i \end{tikzcd}$$

identify $\mathbb{Z}\mathbb{Z}^n \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] = \Lambda$

(i.e. $r_i \in F_n'$) (eg. $G = \pi_1(\mathbb{C}^2 \setminus \cup \mathcal{A})$)

$$\begin{tikzcd} C_*(\tilde{X}): & & \Lambda^m \arrow[r, "d_2"', "= J_G^{ab}"] \arrow[d, dashed, "\varphi"] & \Lambda^n \arrow[r, "d_1"] \arrow[d, equal] & \Lambda \arrow[r, "\varepsilon"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\ \tilde{C}_*(\tilde{T}^n): \arrow[r] & \Lambda^{\binom{n}{3}} \arrow[r, "d_3"] & \Lambda^{\binom{n}{2}} \arrow[r, "d_2"] & \Lambda^n \arrow[r, "d_1"] & \Lambda \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

Then

$$A = \operatorname{coker}(J_G^{ab})$$ Alexander matrix

$$B = \operatorname{coker}\begin{pmatrix} \varphi \\ d_3 \end{pmatrix}$$ mapping cone $\Lambda^m \oplus \Lambda^{\binom{n}{3}} \xrightarrow{\varphi, d_3}$

<u>Exercise</u> (optional) $\varphi_3 = \operatorname{rank}(B/IB)$

Overview of Aomoto (Gelfand) Theory of Hypergeometric Integrals (HG)

① Gauss HG diff. eq.

$$x(1-x)y'' + [c-(1+a+b)x]y' - aby = 0$$

has a solution

$$F[a,b,c,x] = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots$$

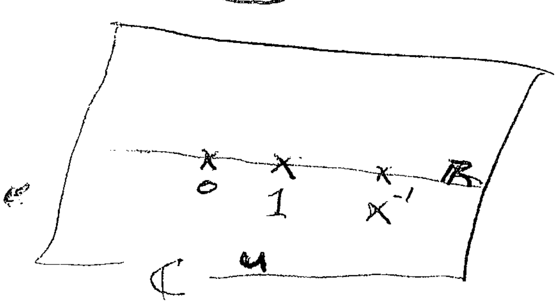
HG series

((when $|x|<1$

$$\frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}\,du$$

$(\operatorname{Re} a>0,\ \operatorname{Re}(c-a)>0\ \ |x|<1)$

② 1-arrangement



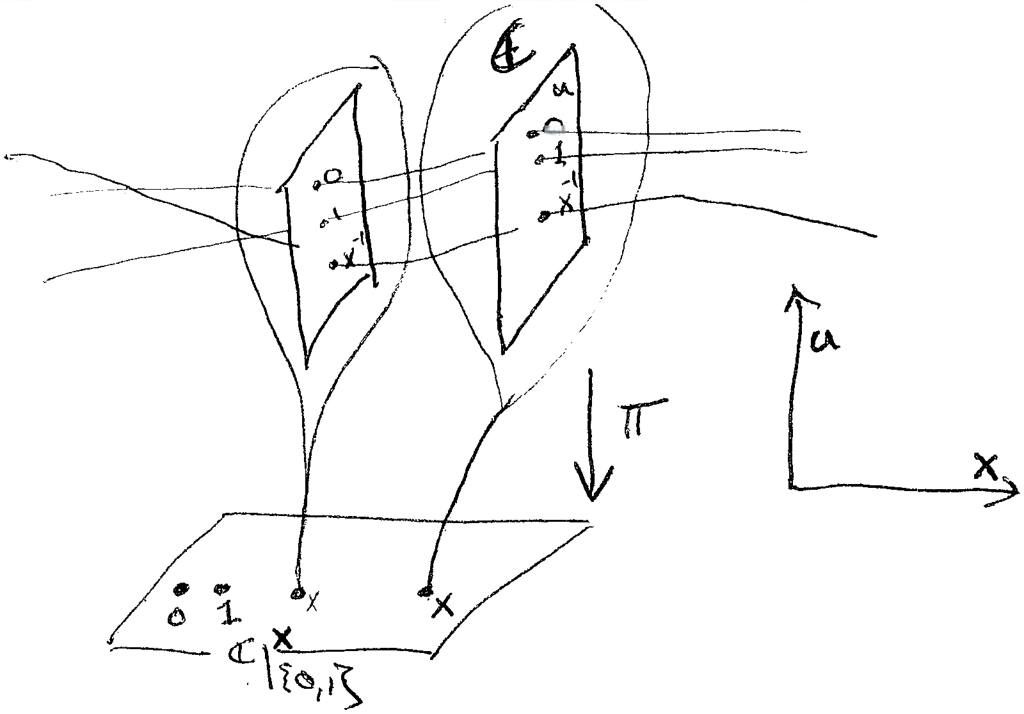
$x \notin \{0,1\}$

parametrized 1-arr.'s

〃

family of 1-arr.'s

combinatorially equiv.



$$\pi^{-1}(x) = M_x = \mathbb{C}\setminus\{0,1,x^{-1}\}$$

$$\Phi(u,\lambda,x) = (1-u)^{\lambda_1} u^{\lambda_2} (1-xu)^{\lambda_3} \qquad \lambda_i \in \mathbb{C}$$

is a multivalued function

$$\omega_\lambda = d\log\Phi(u,\lambda,x) = -\lambda_1 \frac{du}{1-u} + \lambda_2 \frac{du}{u} - \lambda_3 \frac{x\,du}{1-xu}$$

holomorphic funcs sheaf — holo- 1-forms

$$\nabla_x : \mathcal{O}_{M_x} \xrightarrow{\mathbb{C}\text{-linear}} \Omega^1_{M_x}$$

$$f \longmapsto df + f\omega_\lambda$$

$$\operatorname{Ker} \nabla_x = \{ f \in \mathcal{O}_{M_x} \mid df = -f\omega_\lambda \} \overset{\star}{=} \mathbb{C}\left(\frac{1}{\Phi}\right)$$

$$\nabla_x(\Phi^{-1}) = d\left(\frac{1}{\Phi}\right) + \Phi^{-1}\omega_\lambda = -\frac{1}{\Phi^2}d\Phi + \frac{1}{\Phi}d\log\Phi =$$

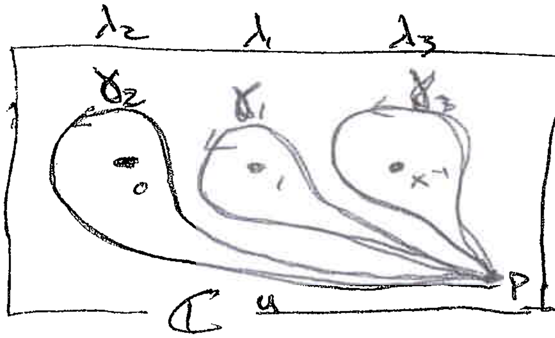
$\mathcal{L}_x := \operatorname{Ker}\nabla_x$ locally constant sheaf

$\mathcal{L}_x$ determines a representation

local system

$$\rho : \pi_1(M_x, P) \longrightarrow \mathrm{Aut}(\mathbb{C}) = \mathbb{C}^*$$

$$\gamma_j \longmapsto \exp(-2\pi i \lambda_j)$$



Consider $\lambda = 0$, then $\Phi = 1$ and $\omega_\lambda = 0$ and

$\nabla = d$. $\mathcal{L}_x = \mathbb{C}$

<u>deRham pairing</u> (nontwisted) non-degenerate pairing

$$H^1(M_x; \mathbb{C}) \times H_1(M_x; \mathbb{C}) \to \mathbb{C}$$

$$[\omega] \in H^1, \quad [\sigma] \in H_1 \longmapsto \int_\sigma \omega$$

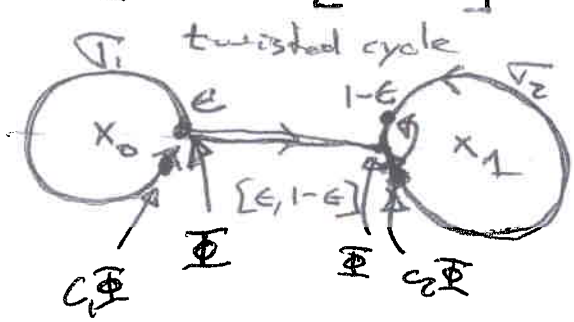
(Stokes) $\displaystyle\int_{\partial\sigma} \omega = \int_\sigma d\omega$

<u>Twisted deRham pairing</u>

$$H^1(M_x; \mathcal{L}_x) \times H_1(M_x; \mathcal{L}_x^\vee) \xrightarrow{\text{non-deg.}} \mathbb{C}$$

dual local system

$[\omega]$ $\quad [\sigma \otimes \Phi]$ $\quad \displaystyle\int_\sigma \Phi\omega$

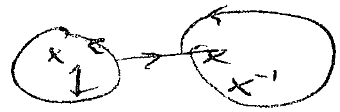


Sps $c_1 \neq 1$ & $c_2 \neq 1$

$$\sigma = \frac{1}{c_1 - 1}\sigma_1 \otimes \Phi + [1-\epsilon, \epsilon] \otimes \Phi - \frac{1}{c_2 - 1}\sigma_2 \otimes \Phi$$

So, $\underline{\partial\sigma = 0}$

$$\partial\sigma = \frac{1}{c_1-1}\left(c_1[\epsilon]\otimes\Phi - [\epsilon]\otimes\Phi\right)$$
$$+\left([1-\epsilon]\otimes\Phi - [\epsilon]\otimes\Phi\right)$$
$$-\frac{1}{c_2-1}\left(c_2[1-\epsilon]\otimes\Phi - [1-\epsilon]\otimes\Phi\right)$$

The other cycle is 

$$\int_{\partial\sigma}\Phi\,\omega = \int_\sigma d(\Phi\omega) = \int_\sigma \Phi\underbrace{(d\log\Phi\wedge\omega + d\omega)}_{\nabla_u\omega}$$

If λ is generic enough
$(\text{eg. } \lambda_1\notin\mathbb{Z},\ \lambda_2\notin\mathbb{Z},\ \lambda_3\notin\mathbb{Z},\text{ and } \lambda_1+\lambda_2+\lambda_3\notin\mathbb{Z})$

$$\dim H^1(M_x, \mathcal{L}_x) = 2$$

oso.

$$\mathbb{A}_1 \curvearrowright \left(\mathbb{C}\frac{du}{1-u}\oplus\mathbb{C}\frac{du}{u}\oplus\mathbb{C}\frac{x\,du}{1-xu}\right)\Big/\nabla 1$$

basis for $H^1(M_x;\mathcal{L}_x)$ is $\begin{cases}\varphi_1 = \lambda_2\dfrac{du}{u}\\ \varphi_2 = -\lambda_3\dfrac{x\,du}{1-xu}\end{cases}$

twisted de Rham

$$0\to\Gamma(M_x,\mathcal{O})\xrightarrow{\nabla}\Gamma(M_x,\Omega^1)\xrightarrow{\nabla}\cdots$$

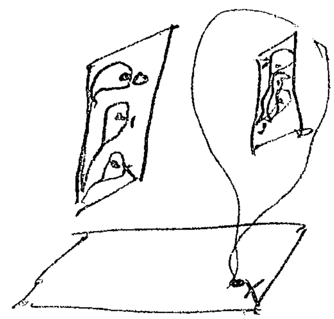
$$H^q(\Gamma(M_x,\Omega^*),\nabla)\cong H^q(M_x,\mathcal{L}_x)$$

$\hat{\varphi}_i = \int_\sigma \Phi \varphi_i \qquad [\sigma] \in H_1(M, \mathcal{L}_x^\vee)$

what kind of diff. equ's are satisfied by $\hat{\varphi}_1, \hat{\varphi}_2$?

diff. equ's $\longleftrightarrow$ cohomologous relations in $H^1(M_x; \mathcal{L}_x)$

d' exterior diff. w.r.t. x



$$d'\hat{\varphi}_1 = d'\int u^{\lambda_1}(1-u)^{\lambda_2}(1-ux)^{\lambda_3}\lambda_2 \frac{du}{u}$$

$$= -\int u^{\lambda_1}(1-u)^{\lambda_2}(1-ux)^{\lambda_3} u \lambda_3 \lambda_2 \frac{du}{u}$$

$$= -\frac{\lambda_2 \cancel{\lambda_3}}{u}\int u^{\lambda_1}(1-u)^{\lambda_2}(1-ux)^{\lambda_3} \underbrace{\frac{\lambda_3 x\, du}{1-xu}}_{= -\varphi_2} dx$$

$$= \frac{\lambda_2 \cancel{\lambda_3}}{x}\hat{\varphi}_2 dx$$

So, $d'\hat{\varphi}_1 = \frac{\lambda_2}{x}\hat{\varphi}_2$

$$d'\begin{bmatrix}\hat{\varphi}_1 \\ \hat{\varphi}_2\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \lambda_3 & \lambda_1+\lambda_3\end{bmatrix}\begin{pmatrix}\hat{\varphi}_1 \\ \hat{\varphi}_2\end{pmatrix}\frac{dx}{x-1} + \begin{bmatrix} 0 & \lambda_2 \\ 0 & -\lambda_1-\lambda_2\end{bmatrix}\begin{bmatrix}\hat{\varphi}_1 \\ \hat{\varphi}_2\end{bmatrix}\frac{dx}{x}$$

To interpret this take

$$\Omega = \begin{pmatrix} 0 & 0 \\ \lambda_3 & \lambda_1+\lambda_3 \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} 0 & \lambda_2 \\ 0 & -\lambda_1-\lambda_2 \end{pmatrix} \frac{dx}{x}$$

log. forms

$$\nabla_\Omega := d' + \Omega \wedge$$

$\begin{bmatrix} \hat{\Phi}_1 \\ \hat{\Phi}_2 \end{bmatrix}$ is a solution for $\nabla_\Omega \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = 0$

Gauss-Manin connection

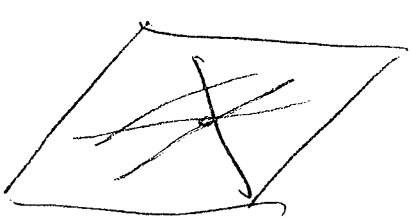
$\frac{d}{dx}\hat{\Phi}_1 = \frac{\lambda_2}{x}\hat{\Phi}_2$ Eliminate $\hat{\Phi}_2$ to have

$$x(1-x)y'' + [(\lambda_1+\lambda_2+1) - (\lambda_2-\lambda_3+1)x]y' + \lambda_2\lambda_3 y = 0$$

has solution $\hat{\Phi}_1$

Set $\begin{cases} \lambda_1 = c-a-1 \\ \lambda_2 = a \\ \lambda_3 = -b \end{cases}$ to recover Gauss H.G. diff. eqn.

③ Higher Dimensional



parametrized combinatorially equ.

generic enough $\lambda = (\lambda_1, \dots, \lambda_n)$ $\quad \alpha_{H_1}^{\lambda_1} \cdots \alpha_{H_n}^{\lambda_n}$

- $H^\ell(M_{\underline{x}}, \mathcal{L}_{\underline{x}})$ has a combinatorially constructed basis independent of x called <u>Bnbc</u> basis $\varphi_1, \dots, \varphi_B$

$\dim H^\ell(M_x, \mathcal{L}_x) = |\chi(M_x)|$

- Ω has logarithmic poles
- ($\mathcal{A}$, complexified real)

  $H_\ell(M_{\underline{x}}, \mathcal{L}_x^\vee)$ has a basis $\sigma_1, \dots, \sigma_B$ corresp. to the set of bdd chambers of $\mathbb{R}^\ell \setminus \bigcup_{H \in \mathcal{A}} (H \cap \mathbb{R}^\ell)$

- $PM(\mathcal{A}, \underline{x}) = [\langle \varphi_i, \sigma_j \rangle]$

Varchenko's formula for $\det PM(\mathcal{A}, \underline{x})$ using the "B-fun" of $\mathcal{A}$

$B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}\,du$.

Lecture 11 — Yuzvinsky

① A OS.

$$0 \to A_0 \xrightarrow{\cdot a} A_1 \xrightarrow{\cdot a} \cdots \xrightarrow{\cdot a} A_\ell \to 0$$

Take $a \in A_1 = E_1$; $a^2 = 0$

$$\boxed{H^*(A, a) = ?}$$

For Affine

$$I = \langle de_S, e_T \mid S = \text{circuit}, \bigcap_{H \in T} H = \emptyset \rangle$$

$$A = \langle e_1, \dots, e_n \rangle = \left\langle \left[\frac{d\alpha_1}{\alpha_1}\right], \dots, \left[\frac{d\alpha_n}{\alpha_n}\right] \right\rangle$$

② Motivations 1) $a \in E_1$, a is regular on A if $H^*(A, a) = 0$

otherwise a is singular

2) Relations to local Coefficients.

$$\begin{array}{ccccccccccc} 0 \to & \Omega^0 & \xrightarrow{d + w_a \wedge} & \Omega^1 & \xrightarrow{d + w_a \wedge} & \cdots & \xrightarrow{d + w_a \wedge} & \Omega^\ell & \xrightarrow{d + w_a \wedge} & 0 \\ & \cup & & \cup & & & & \cup & & \\ 0 \to & A^0 & \longrightarrow & A^1 & \to & A^2 & \to \cdots \to & A^\ell & & \\ & & & \ni a & & & & & & \end{array}$$

$w \in \Omega^1$ under the embedding

functions

$\sum c_{i_1 \dots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$

skew comm. rule

$$a = w_a = \sum a_i e_i = \sum a_i \frac{d\alpha_i}{\alpha_i}$$

$$(d + w_a \wedge) \frac{d\alpha}{\alpha} = w_a \wedge \frac{d\alpha_i}{\alpha_i} = a \cdot e_i$$

$$(d + w_a \wedge) w_b = a \cdot b$$

$b \in A$

Define; $\mathcal{A}$ = Boolean $\Rightarrow$ $i^{\cdot}$ is quasi-iso.

if $a_i \notin \mathbb{N}$ $\Rightarrow$ i^* is iso.

doesn't make sense

Thm [ESV] + [STV']

(Proj. Arr.) $i^{\cdot}$ is quasi-iso if

$$\sum_{H_i \supset X} a_i \notin \mathbb{N} \quad \forall X \in L \Rightarrow \mathcal{A}_X \text{ is irred.}$$

3) "Generic" $a \in A_1 \rightsquigarrow$ (vanishing)

a) Central, $\sum_{i=1}^{n} a_i \neq 0 \Rightarrow H^*(A, a) = 0$

$$\begin{tikzcd} A_{p-1} \arrow[r, "a"] \arrow[dl, dashed, "h"'] & A_p \arrow[r, "a"] \arrow[dl, dashed, "h"'] & A_{p+1} \arrow[dl, dashed] \\ A_{p-1} \arrow[r, "a"] & A_p \arrow[r, "a"] & A_{p+1} \end{tikzcd}$$

If $ha + ah = 1$ then $H^*(A, a) = 0$.

Here $h = d = \sum (-1)^{j-1} e_{S_j}$

Pr 1) Well-defined 2) Prove $d \cdot a + a \cdot d = \sum_{i=1}^{n} a_i$

b) Thm (i) $\sum_{H_p \supset X} a_i \neq 0 \quad \forall X \in L \Rightarrow H^p(A, a) = 0 \quad \forall p < \ell$

(suffices $\mathcal{A}_X$ is irred.)

(ii) $\mathcal{A}$ is central $\sum_{H_i \supset X} a_i \neq 0 \quad \forall X \in L \setminus \{0\}$

$\Rightarrow H^p(A, a) = 0 \quad p < \ell - 1 \quad \dim H^{\ell-1} = \dim H^\ell = |\chi(M(d\mathcal{A}))|$

if complexified real $=$ # of bdd chambers

$= \beta$-invariant

Def. If $R^p_d = \{a \in A_1 \mid \dim H^p(A,a) \geq d\}$

resonance variety

Pr 2 Prove that R^p_d is affine subvar. of A_1 $(\cong \mathbb{C}^n)$.

(4) Propagation of cohom.

Thm If $H^p(A,a) \neq 0$ then $H^q(A,a) \neq 0$ for $p \leq q \leq l$. (central, essential)

Combinatorial Structure of Arrangements

$\mathcal{A} = \{H_1, \ldots, H_n\}$ hyp. arr. in $V = \mathbb{K}^\ell$, $\mathbb{K}$-field

$\alpha_i \in V^*$ w/ $\ker \alpha_i = H_i$.

Properties of minimal ~~in~~dependent sets (Whitney, Maclane)

Let $\mathcal{C}$ = set of minimal dep. subsets of $\{\alpha_1, \ldots, \alpha_n\} \subseteq V^*$ "circuits"

$\Rightarrow$ [$C \in \mathcal{C} \Leftrightarrow \sum_{i \in C} c_i \alpha_i = 0$ w/ $c_i \neq 0$ $\forall i$]

(i) $\emptyset \notin \mathcal{C}$

(ii) $A \in \mathcal{C}$ and $B \supsetneq A$ then $B \notin \mathcal{C}$

(iii) if $A, B \in \mathcal{C}$ and $e \in A \cap B$ then
$\exists\, C \in \mathcal{C}$ with $C \subseteq A \cup B - \{e\}$

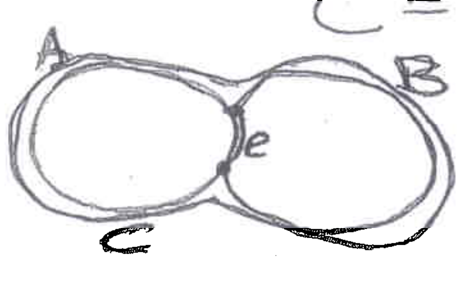
Def: A <u>matroid</u> w/ ground set $[n] = \{1, \ldots, n\}$ or E is (determined by) a set $\mathcal{C} \subseteq 2^{[n]}$ or 2^E satisfying (i)-(iii)

Ex: Let Γ be a finite graph, $E = E(\Gamma)$ the set of edges.

Let $\mathcal{C} = \{C \subseteq E \mid C \text{ is a cycle (or circuit)}\}$

$C \simeq$  $\Big\}$ Graphic Matroid

Here's (iii)



Graphic Arrangements Γ = graph w/o loops or multiple edges

$$\mathcal{A}_\Gamma = \{H_{ij} \mid ij \in E(\Gamma)\} \quad H_{ij} = \ker(x_i - x_j)$$

circuit in $\mathcal{A}_\Gamma \Longleftrightarrow$ circuit in Γ

Note: Graphic arr's $\longleftrightarrow$ subarr's of the braid ar

$\updownarrow$

Graphic matroids

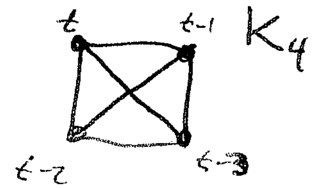
chromatic polynomial

$\chi_\Gamma(n)$ = # of colorings of vertices of Γ w/ n colors and adjacent vertices having different colors

change of variables

$\mathrm{Poin}(\mathcal{A}_\Gamma, t)$

Ex: Γ



$\mathcal{A}_\Gamma$ = braid arr. $= \mathcal{A}_4$

$$\chi_\Gamma(t) = t(t-1)(t-2)(t-3)$$

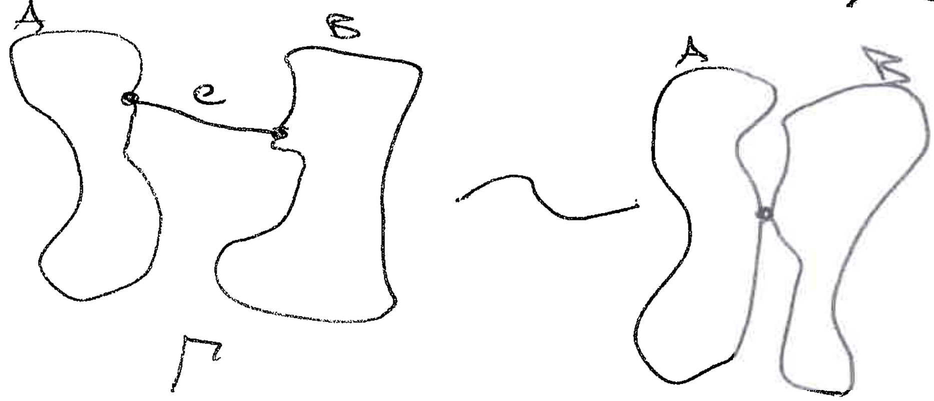
$$\mathrm{Poin}(\mathcal{A}_4, t) = (1+0t)(1+t)(1+2t)(1+3t)$$

$$\mathrm{Poin}(\mathcal{A}_\Gamma, t) = (-t)^{r(\Gamma)}\chi_\Gamma\left(-\tfrac{1}{t}\right)$$

deletion of $e = ij$ from Γ $= \Gamma - e$

$\longleftrightarrow$ deletion of H_{ij} from $\mathcal{A}_\Gamma$, $\mathcal{A}_\Gamma'$

contraction of e in Γ $= \Gamma/e$



$\longleftrightarrow$ contraction or restriction $\mathcal{A}_\Gamma''$ of H_{ij} in $\mathcal{A}_\Gamma$ in $(\mathcal{A}_\Gamma, t)$

<u>Thm</u>: $\chi_\Gamma(n) = \chi_{\Gamma - e}(n) - \chi_{\Gamma/e}(n)$

<u>Pf</u>: "different = all − same"

<u>Cor.</u>: $\mathrm{Poin}(\mathcal{A}_\Gamma, t) = \mathrm{Poin}(\mathcal{A}_\Gamma', t) + t\,\mathrm{Poin}(\mathcal{A}_\Gamma'', t)$

<u>Note</u>: This follows from $0 \to A' \to A \to A''(-1) \to 0$

<u>Remark</u>: $\exists$ many equivalent axiomatizations of matroids in terms of:

$\left.\begin{array}{l}\text{dependent sets}\\ \text{indep. sets}\\ \text{maximal indep. sets}\\ \text{closed sets}\\ \text{closure operator}\\ \text{"hyperplanes" or "bonds"}\end{array}\right\}$ cryptomorphisms

or 30 others

Closed sets Let G be a matroid on $[n]$. $S \subseteq [n]$

The closure, $\bar{S}$, of S is defined by

$$i \in \bar{S} \iff \exists \text{ circuit } C \in G \text{ with } i \in C \text{ and } |C \cap S| \geq |C| - 1$$

S is closed iff $\bar{S} = S$

closed sets $\equiv$ "flats"

The collection of closed sets $L(G)$ is a geom. la

If G is the underlying matroid of $\mathcal{A}$ then $L(G) \cong L(\mathcal{A})$.

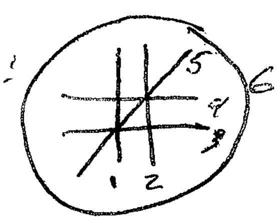
Pictures

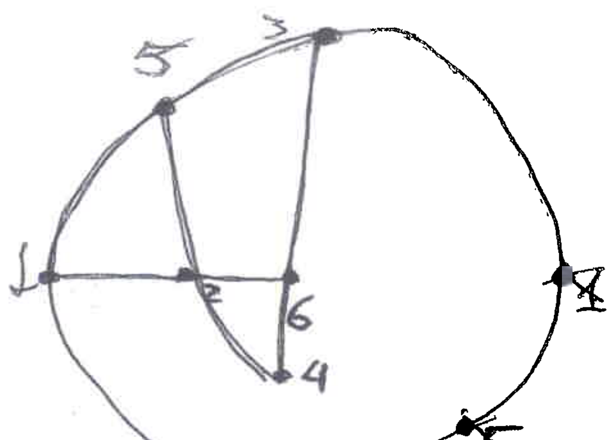
hyp. arr.'s in $\underset{V}{\underset{\shortparallel}{\mathbb{K}^\ell}}$ $\longleftrightarrow$ point config.'s in $\mathbb{P}(V^*)$

$\{H_1, \ldots, H_n\} \longleftrightarrow \{\alpha_1, \ldots, \alpha_n\}$

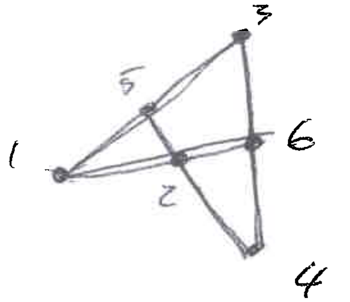
normal lines

Ex:

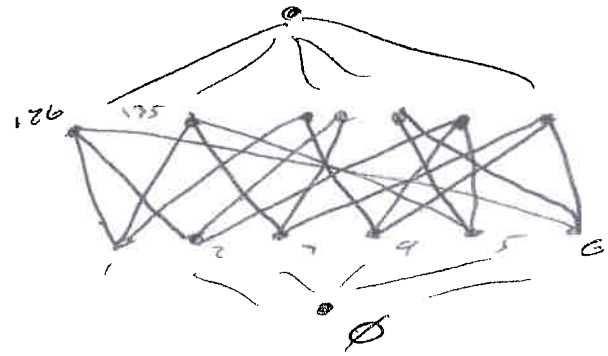




$H_1 = X$ $\alpha_1 = [1\ 0\ 0]$
$H_2 = X - Z$ $\alpha_2 = [1\ 0\ -1]$
$H_3 = Y$ $\alpha_3 = [0\ 1\ 0]$
$H_4 = Y - Z$ $\alpha_4 = [0\ 1\ -1]$
$H_5 = X - Y$ $\alpha_5 = [1\ -1\ 0]$
$H_6 = Z$ $\alpha_6 = [0\ 0\ 1]$

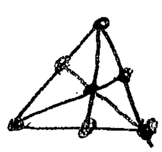


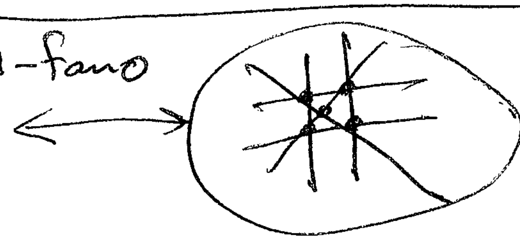
135 is a flat

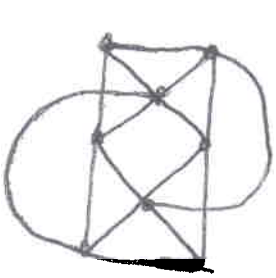


Other Examples of Matroids

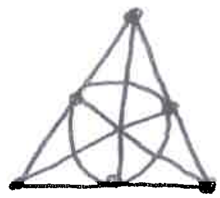
non-fano







MacLane



fano

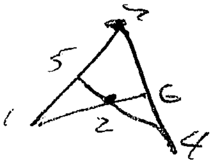
Realization Spaces

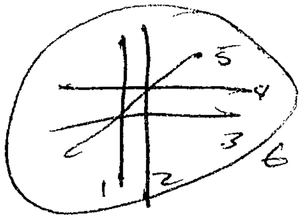
Problem: Given a matroid G and a field $\mathbb{K}$, construct a parametrization of the set of all labeled proj. pt config.'s in $\mathbb{P}(\mathbb{K}^2)$ w/ matroid G, up to proj. equiv.

Ex 1 "the braid arr. $\mathcal{A}_4$ is proj. unique"

Method: Choose a proj. basis B in G

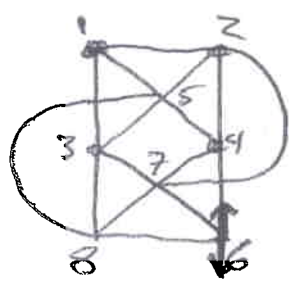
B has $B = \{i_1, i_2, i_3, i_4\}$ with each triple $\subseteq B$ independent.

Ex:  $B = 1234$



Any realization of G is proj. eq. to a unique realization of the form

$$\begin{array}{r c}
1 & \\
2 & \\
3 & \\
4 & \\
(1\vee 2)\wedge(3\vee 4) = 5 & \\
\approx 6 &
\end{array}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}$$

Ex: 

$$\begin{array}{r}
1 \\
2 \\
3 \\
4 \\
(1\vee 4)\wedge(2\vee 3) = 5 \\
(3\vee 6)\wedge(1\vee 2) = 6 \\
= 7 \\
(1\vee 3)\wedge(4\vee 7) = 8 \\
\text{and } (5\vee 6)
\end{array}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & t & 1 \\
1 & t & 0 \\
t-1 & 0 & t
\end{bmatrix}
\qquad t \neq 1$$

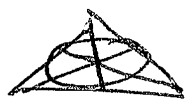
$$\det(568) = 0 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & t & 1 \\ t-1 & 0 & t \end{vmatrix} = t^2 + t + 1$$

So, $t = \omega, \omega'$ where $\omega^3 =$

Conclusions:

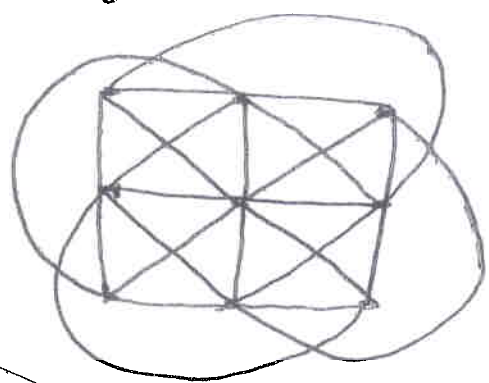
- $\exists$ 2 inequivalent realizations over $\mathbb{C}$
- $\nexists$ realizations over $\mathbb{R}$
- $\exists$ realizations over $\mathbb{K}$ if $\mathrm{char}(\mathbb{K})=3$

Exercises

(1) Show  is realizable only over fields of $\mathrm{char}(\mathbb{K})=2$.

(2) Is the arr. $Q(\mathcal{A})=(x^m-y^m)(x^m-z^m)(y^m-z^m)$ proj. unique over $\mathbb{C}$

$m=3$



Ref.

Bokowski & Sturmfels Comp. Synth. Geom.

Lecture 13 Suciu

Characteristic Varieties of Arr's

§0 Intro

$\mathcal{A}$ arr. $|\mathcal{A}|=n$, $M(\mathcal{A})$: complement

$G=\pi_1(M(\mathcal{A}))$ $\mathbb{K}$ - field (alg. closed $\mathbb{K}=\mathbb{C}$)

Char. Varieties of $\mathcal{A}$ over $\mathbb{K}$

$$V_d^i(\mathcal{A},\mathbb{K}) := \{t\in(\mathbb{K}^*)^n \mid \dim_{\mathbb{K}} H^i(M,\mathbb{K}_t)\geq d\}$$

$\mathbb{K}_t$: rank 1 local system defined by t

$$(\mathbb{K}^*)^n = V_0^i \supset V_1^i \supset V_2^i \supset \cdots \supset V_n^i$$

Resonance Varieties

$$R_d^i(\mathcal{A},\mathbb{K}) = \{a\in\mathbb{K}^n \mid \dim_{\mathbb{K}} H^i(H^*(M,\mathbb{K}),\cdot a)\geq d\}$$

Object of the talk:

- interpret $V_d(\mathcal{A}) := V_d^1(\mathcal{A},\mathbb{K})$ and $R_d(\mathcal{A}) := R_d^1(\mathcal{A})$ in concrete te
- relate these two varieties
- Applications
  - homology of finite cover
  - Milnor fibration.

- homology of Milnor fiber
- # of epi's to finite grps
- counting finite index sbgps

1. Homology of finite Cover

$$\xrightarrow{\varphi\;}\!\!\!\!\rightarrow G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle \;, \quad r_i \in [F_n, F_n]$$

$\dots, x_n\rangle$

X = 2-complex modelled in this presentation

$\tilde{X}$ = universal cover

$$\tilde{C}_*(\tilde{X}): \quad \begin{array}{ccccccccc} C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\varepsilon} & \mathbb{Z} & \to & 0 \\ \| & & \| & & \| & & \| & & \\ (\mathbb{Z}G)^m & \xrightarrow[J_G^{ab} = \left(\frac{\partial r_i}{\partial x_j}\right)]{} & (\mathbb{Z}G)^n & \xrightarrow[\begin{pmatrix} x_1 - 1 \\ \vdots \\ x_n - 1 \end{pmatrix}]{} & \mathbb{Z}G & \to & \mathbb{Z} & \to & 0 \end{array}$$

$$\overset{\text{Sps}}{1 \to K_\gamma \hookrightarrow G \xrightarrow{\gamma}\!\!\!\!\rightarrow \Gamma \to 1} \quad \Longleftrightarrow \quad \begin{array}{c} \Gamma \to X^\gamma \\ \downarrow \\ X \end{array}$$

reg. Γ-cover with $\pi_1(X^\gamma) = K_\gamma$

We have:

$$C_*(\tilde{X}^\gamma): \quad \begin{array}{ccccc} C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \\ \| & & \| & & \| \\ (\mathbb{Z}\Gamma)^m & \xrightarrow[J_G^\gamma]{} & (\mathbb{Z}\Gamma)^n & \xrightarrow[\begin{pmatrix} \gamma(x_1) - 1 \\ \vdots \\ \gamma(x_n) - 1 \end{pmatrix}]{} & \mathbb{Z}\Gamma \end{array} \qquad (*)$$

ex: $1 \to G' \to G \xrightarrow{ab} \mathbb{Z}^n \to 1$ $\quad X^\gamma = X^{ab}$ ab. cov

Λ = Laurent poly $= \mathbb{Z}\mathbb{Z}^n$

$$C_*(X^{ab}): \quad \Lambda^m \xrightarrow{M = J_G^{ab}} \Lambda^n \xrightarrow{\begin{pmatrix} t_1 - 1 \\ \vdots \\ t_n - 1 \end{pmatrix}} \Lambda$$

M = alex. matrix

ex: Γ finite, $|\Gamma| = k$,

(*) becomes

$$\mathbb{Z}^{km} \xrightarrow[J^\gamma]{} \mathbb{Z}^{kn} \xrightarrow[\partial_1^\gamma]{} \mathbb{Z}^k$$

where

$$G \longrightarrow \mathrm{Sym}\Big(\overset{\Gamma}{\overset{\parallel}{G/K_\gamma}}\Big) \cong S_k \hookrightarrow GL(k, \mathbb{Z})$$

coset repr. $\qquad$ perm. rep.

Problem 1: $H_1(K_\gamma) \oplus \mathbb{Z}^{k-1} \cong \operatorname{coker}(J^\gamma)$

Now, sps $\mathbb{K}$ is sufficiently large w.r.t. Γ (either $\operatorname{char}\mathbb{K} = 0$ or $\operatorname{char}\mathbb{K} \nmid |\Gamma|$

Then: $\dim_{\mathbb{K}} H_1(K_\gamma, \mathbb{K}) = n + \sum\limits_{\substack{\rho \in \mathrm{Irr.\,rep} \\ \rho \neq 1}} n_\rho (\operatorname{corank} J^{\rho \circ \gamma} - n_\rho)$

where $n_\rho = \dim \rho$

$\rho : \Gamma \longrightarrow GL(n_\rho, \mathbb{K})$

2. Characteristic Varieties of G

$$\hat{G} = \mathrm{Hom}(G, \mathbb{K}^*) \cong \mathrm{Hom}(\mathbb{Z}^n, \mathbb{K}^*) \cong (\mathbb{K}^*)^n$$

character torus

$G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$

$(\mathbb{K}^*)^n \ni t = (t_1, \dots, t_n) \longleftrightarrow \mathbb{K}_t^* =$ rank 1 ~~vector~~ local system

i.e. $\mathbb{K}$ viewed as a $\mathbb{Z}G$-mod.

$$G \xrightarrow{ab} \mathbb{Z}^n \longrightarrow \mathbb{K}^* = GL(1, \mathbb{K})$$

i^{th} basis element $\longmapsto t_i$

$C_*(X, \mathbb{K}_t) = C_*(\hat{X}) \otimes_{\mathbb{Z}G} \mathbb{K}_t$:

which gives

$$\mathbb{K}^m \xrightarrow[\quad]{\partial_2^t} \mathbb{K}^n \xrightarrow[\quad]{\partial_1^t} \mathbb{K}$$

$\partial_2^t = M(t)$ = Aless. matrix evaluated at t

$\partial_1^t = \begin{pmatrix} t_1 - 1 \\ \vdots \\ t_n - 1 \end{pmatrix}$

Then $H_1(G, \mathbb{K}_t) = H_1(X, \mathbb{K}_t) = \ker(\partial_1^t) / \mathrm{Im}(\partial_2^t)$

__Remark__ $H^1(G; \mathbb{K}_t) = H_1(G, \mathbb{K}_{t^{-1}})$

Def: $V_d(G,\mathbb{k}) = \{ t \in (\mathbb{k}^*)^n \mid \dim_{\mathbb{k}} H^1(G, \mathbb{k}_t) \geq d \}$

From the above discussion we get:

$$\boxed{V_d = \{ t \mid \operatorname{rank} M(t) < n-d \}}$$

= zero locus of ideal of codim d minor of Alex. matrix specialized at t.

— a subvariety of char. torus

Problem 2: If $G = \pi_1(M(\mathcal{A}))$, then

$$V_d(G) = V_d(\mathcal{A}) \text{ and } R_d(G) = R_d(\mathcal{A})$$

[Hint: Use the fact that $\underset{\text{Hurewicz}}{h}: \pi_2 M \to H_2 M$ is zero + Hopf exact seq. $\pi_2 X \to H_2 X \to H_2 G \to 0$

Problem 3: Show that all entries in the Alex. matrix of an arr. lie in the augmentation ideal of $\Lambda = \mathbb{Z}\mathbb{Z}'$

Ex

$\mathcal{A} =$  $\qquad G = (x_1, x_2, x_3 \mid x_1x_2x_3 = x_3x_1x_2 = x_2x_3x_1) \cong F_1 \times F_2$

$$M = J_G^{ab} = \begin{bmatrix} 1-t_3 & t_1 - t_3t_2 & t_1t_2 - 1 \\ 1-t_2t_3 & t_1 - 1 & t_1t_2 - t_2 \end{bmatrix}$$

$V_1(G)$ = zero set of 2×2-minors

1^{st} minor $= (1-t_3)(t_1-1) - t_1(1-t_3)(1-t_2t_3) = (1-t_3)(t_1 - 1 - t_1 + t_1t_2t_3)$

2^{nd} minor $= (t_2-1)(1+t_3-t_2)$

3^{rd} minor $= (t_1-1)(1-t_1t_2t_3)$

$$V_1(G) = \{t \in (\mathbb{K}^*)^n \mid t_1t_2t_3 - 1 = 0\}$$
$$V_2(G) = \{1\}$$

Problem 4:  $\}$ n lines

$G \cong F_{n-1} \times F_1$

$$V_1 = \{t \in (\mathbb{K}^*)^n \mid t_1 \cdots t_n - 1 = 0\}$$
$$\parallel$$
$$V_2 = \cdots = V_{n-1} \qquad V_n = \{1\}$$

Problem 5

$$V_d(\mathcal{A}) = \left\{ t \in (\mathbb{K}^*)^n \;\middle|\; \begin{array}{l}(t_1, \ldots, t_{n-1}) \in V_d(d\mathcal{A}) \\ \& \; t_1 \cdots t_n - 1 = 0\end{array} \right\}$$

General description cv's of $\mathcal{A}$'s (for $\mathbb{K} = \mathbb{C}$)

$$V_d(\mathcal{A}) = \bigcup_{i=1}^{s} \rho_i T_i$$

where $T_i = (\mathbb{C}^*)^{r_i}$ is a subtorus of $T = (\mathbb{C}^*)^n$

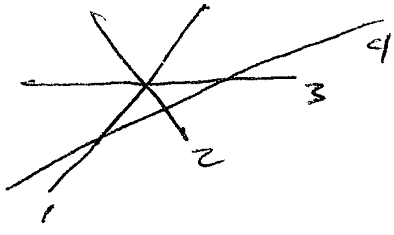
ρ_i has finite order $(\rho_i^{k_i} = 1)$

This Follows from work of – Green-L.

Simpson

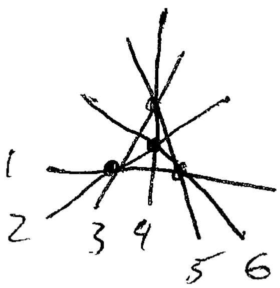
– Arapura

More examples



$$V_1 = \{ t \in (\mathbb{C}^*)^4 \mid t_1 t_2 t_3 = 1,\ t_4 = 1 \}$$

is a 2-dim. Torus



$$V_1 = \{ t \in (\mathbb{C}^*)^6 \mid t_1 t_2 t_3 = t_4 = t_5 = t_6 = 1 \} \cup$$

$$\{ t \mid t_2 t_4 t_6 = t_1 = t_3 = t_5 = 1 \} \cup$$

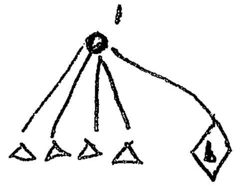
$$\{ t \mid t_3 t_4 t_5 = t_1 = t_2 = t_6 = 1 \} \cup$$

$$\{ t \mid t_1 t_5 t_6 = t_2 = t_3 = t_4 = 1 \} \cup$$

$$\{ t \mid t_1 = t_4,\ t_2 = t_5,\ t_3 = t_6,\ t_1 \cdots t_6 = 1 \}$$

$$\cancel{P_1 \cup P_2 \cup P_3} = T_1 \cup T_2 \cup T_3 \cup T$$

V_2



Connection to resonance varieties

$$R_d(G) = \{ a \in \mathbb{k}^n \mid \dim_{\mathbb{k}} H^1(H^*(G, \mathbb{k}), \cdot a) \geq d \}$$

$$= \{ a \in \mathbb{k}^n \mid \operatorname{rank}_{\mathbb{k}} M^{\mathrm{lin}}(a) < n - d \}$$

where M^{lin} = linearized Alex. matrix

$t_i \longrightarrow 1 - a_i$ then take linear

$t_i^{-1} \longrightarrow 1 + a_i + a_i^2 + \cdots$ terms

This is now a matrix of linear form

e.g. $\mathcal{A} = \times$

$$M = \begin{bmatrix} 1 - t_3 & t_1(1-t_3) & t_1 t_2 - 1 \\ 1 - t_2 t_3 & t_1 - 1 & t_2(t_1 - 1) \end{bmatrix}$$

$$M^{lin} = \begin{bmatrix} a_3 & a_3 & -a_1 - a_2 \\ a_2 + a_3 & -a_1 & -a_1 \end{bmatrix}$$

$$R_1 = \{a \in \mathbb{K}^3 \mid a_1 + a_2 + a_3 = 0\}$$

Theorem:

tangent cone

$$T_1(V_d(\mathcal{A}, \mathbb{C})) = R_d(\mathcal{A}, \mathbb{C})$$

in particular $R_d(\mathcal{A}, \mathbb{C})$ is a union of subspaces in $\mathbb{C}^n$.

Comments: • $T_1(V_d^i(\mathcal{A}, \mathbb{C})) = R_d^i(\mathcal{A}, \mathbb{C})$

• $T_1(V_d(G, \mathbb{C})) \subseteq R_d(G, \mathbb{C})$ $\neq$ in general

• $R_d(\mathcal{A}, \mathbb{K})$ need not be linear if char $k > 0$

•

Lecture 14 Falk

I. Genericity Conditions

(ESV + STV)

$\mathcal{A}$: essential affine arr. in $\mathbb{C}^\ell$

$\lambda \in \mathbb{C}^n \longleftrightarrow t = \exp(2\pi i \lambda) \in (\mathbb{C}^*)^n$

$a_\lambda = \sum_{i=1}^{n} \lambda_i a_i$ $\quad \mathcal{L}_t$ = rank 1 local system on M determined by t

$$X \in L \longrightarrow \lambda_X = \sum_{H_i \supseteq X} \lambda_i$$

(ESV + STV) If $\lambda_X \notin \mathbb{Z}_{>0}$ for every irreducible $X \in L(\mathcal{A}_\infty)$, $X \neq 0$ (i.e. $X \notin L(\mathcal{A})$, $X \neq 1_L$) then $H^*(M, \mathcal{L}_t) = H^*(A, a_\lambda)$

$\mathcal{A}_\infty = \mathcal{A} \cup \{H_\infty\}$

H_∞

$$\lambda_\infty := -\sum_{i=1}^{n} \lambda_i$$

($Y_{\mathcal{A}}$) If $\lambda_X \neq 0 \ \forall X \in L(\mathcal{A})$ then

$$H^k(A, a_\lambda) = \begin{cases} 0 & k \neq \ell \\ \mathbb{C}^{\beta(\mathcal{A})} & k = \ell \end{cases}$$

$\beta(\mathcal{A})$ = beta invariant of $c\mathcal{A}$

$$= \left| \frac{\mathrm{Poin}(L(c\mathcal{A}), t)}{(1+t)} \Big|_{t=-1} \right|$$

$= |\chi(M(\mathcal{A}))| \underset{\text{Zaslavsky}}{\overset{\text{Thm}}{=}}$ # Bdd Chambers of $M(\mathcal{A}) \cap \mathbb{R}^\ell$ if $\mathcal{A}$ is ~~complexified~~ real

II. "Gauge Transformations"

① $\lambda, \lambda' \in \mathbb{C}^n$ and $\lambda - \lambda' \in \mathbb{Z}^n$

$\Rightarrow \mathcal{L}_t = \mathcal{L}_{t'} \Rightarrow H^*(M, \mathcal{L}_t) = H^*(M, \mathcal{L}_{t'})$

But $H^*(A, a_\lambda)$ & $H^*(A, a_{\lambda'})$ are not necessarily ~~equal~~ iso.

② If $\lambda' = c\lambda$, $c \in \mathbb{C}^*$ then $H^*(A, a_\lambda) \cong H^*(A, a_{\lambda'})$

but $H^*(M, \mathcal{L}_t)$ & $H^*(M, \mathcal{L}_{t'})$ are not necessarily ~~equal~~ iso.

Note: (1) each H_i is irreducible

(2) $rk(X)=2 \Rightarrow$, X is irreducible iff $|\mathcal{A}_X| \geq 3$.

Modifications of λ:

Assume $\lambda_i \notin \mathbb{Z}$ $\forall i$ and $\sum_{i=1}^{n} \lambda_i \notin \mathbb{Z}$

Replace λ_i with $\underset{\lambda_i'}{\underbrace{\lambda_i - N}}$, $N \gg 0$, $N \in \mathbb{Z}$

for $i=1,\ldots,n$

Then $\lambda_X' \notin \mathbb{Z}_{>0}$ for any $X \not\subseteq H_\infty$

$$\lambda_\infty' = \left(-\sum_{i=1}^{n} \lambda_i\right) + nN$$

If no $X \subseteq H_\infty$, $X \neq H_\infty$, is irr., then ESV/STV cond. is satisfied.

$$H^*(M, \mathcal{L}_t) \cong H^*(A, a_{\lambda'})$$

Degree One Resonance Varieties

Assume $\ell \geq 2$

For what λ is $H^1(A, a_\lambda) \neq 0$?

$\mathcal{R}^1(\mathcal{A}) = \{\lambda \in \mathbb{C}^n \mid H^1(A, a_\lambda) \neq 0\}$ $\mathbb{K}$ any field

$\mathcal{R}^1_d(\mathcal{A}) = \{\lambda \in \mathbb{C}^n \mid \dim_{\mathbb{K}} H^1(A, a_\lambda) \geq d\}$

$$0 \to A_0 \xrightarrow{a_\lambda} A_1 \xrightarrow{a_\lambda} A_2 \xrightarrow{a_\lambda} \cdots$$
$$A_0 = \mathbb{K}$$

$a_\mu \in A_1$ represents a non-trivial element of $H^1(A, a_\lambda)$
iff $a_\lambda \wedge a_\mu = 0$ and $a_\mu \notin \mathbb{K} a_\lambda$
(i.e. $\mu \notin \mathbb{K}\lambda$)

Lifting to E:

$$0 \neq e_\lambda \wedge e_\mu \in I^2 \iff 0 \neq [a_\mu] \in H^1(A, a_\lambda)$$

Recall: Assume $\mathcal{A}$ is central (of rank 3)

$$A_2 = \bigoplus_{\substack{X \in L \\ rk(X)=2}} A_X^2$$

Write $a_\lambda^X = \sum_{H_i \geq X} \lambda_i a_i$

Then $a_\lambda \wedge a_\mu$ iff $a_\lambda^X \wedge a_\mu^X = 0$ $\forall X \in L, rk(X) = 2$

Problem 1: Sps $\mathcal{A}$ has rk two. Show

$a_\lambda \wedge a_\mu = 0$ iff $\mu \in \mathbb{K}\lambda$ or $\underbrace{\sum_{i=1}^{n} \lambda_i = 0}_{n \geq 3} = \sum_{i=1}^{n} \mu_i$ and

(Hint: Recall $\partial e_{ijk} = (e_i - e_j) \wedge (e_j - e_k)$)

Thm: $\lambda \in \mathcal{R}^1(A, \mathbb{K})$ iff $\exists\, \mu \in \mathbb{K}^n$ s.t. $\forall X \in L(\mathcal{A}), rk(X) = 2$

either (i) $\mu^X \parallel \lambda^X$ and $\lambda \nparallel \mu$

or (ii) $|\mathcal{A}_X| \geq 3$ and $\lambda_X = \mu_X = 0$.

$$\lambda^X = (\lambda_i \mid H_i \in \mathcal{A}_X) \in \mathbb{K}^{|\mathcal{A}_X|}$$

Assume λ & μ satisfy theorem then we say (λ, μ) is a <u>resonant pair</u>.

Define $\operatorname{supp}(\lambda,\mu) = \{ i \in [n] \mid \lambda_i \neq 0 \text{ or } \mu_i \neq 0 \}$

Assume $\operatorname{supp}(\lambda,\mu) = [n]$

Define an equivalence relation on $[n]$ by $i \sim j$ iff $\begin{vmatrix} \lambda_i & \mu_i \\ \lambda_j & \mu_j \end{vmatrix} = 0$.

The Theorem now says:

$\forall X \in L$, $\operatorname{rk}(X) = 2$ either

(i) $i \sim j \ \forall i,j \in X$

or (ii) $|X| \geq 3$ and $\lambda_X = 0 = \mu_X$

<u>Lemma</u>: Sps $\mathcal{A}$ has rk two. Then A_2 has ~~bas~~ basis

$$\{ a_1 \wedge a_j \mid 2 \leq j \leq n \}$$

<u>Pf</u>: Exercise

Thm: Sps $X = \{1, \ldots, p\}$ and $\operatorname{rk} X = 2$ (88)
$= \{1, \ldots, p\}$

$i \sim j \quad \forall \quad i, j \geq 2$

Then $\quad i \sim j \quad \forall i, j \geq 1$

Pf: $0 = \underbrace{\left(\sum_{i=1}^{p} \lambda_i a_i\right)}_{a_\lambda^X} \wedge \underbrace{\left(\sum_{j=1}^{p} \mu_j a_j\right)}_{a_\mu^X} = \sum \begin{vmatrix} \lambda_i & \mu_i \\ \lambda_j & \mu_j \end{vmatrix} a_i \wedge a_j = 0$

$$= \sum_j \begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_j & \mu_j \end{vmatrix} a_1 \wedge a_j$$

By lemma $\begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_1 & \mu_1 \end{vmatrix} = 0 \quad \forall j.$

Def: A partition P of $[n]$ is neighborly (w.r.t. $\mathcal{A}$) iff for every block B of P and " " " flat X of $\operatorname{rk} X = 2$ $|X \cap B| \geq |X| - 1$ then $X \subseteq B$

Cor. If (λ, μ) is a resonant pair, then the induced partition of $[n]$ is neighborly.

Cor. If (λ, μ) is a resonant pair and X is a flat of rk 2 with $X \nsubseteq$ any block of the induced partition then $\lambda_X = 0 = \mu_X$.

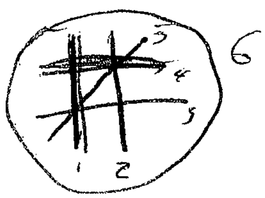
So, we have:

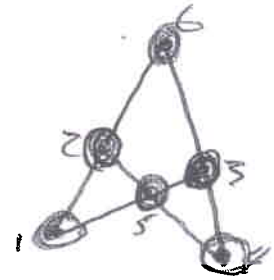
If ~~X∈B'~~ $\mathcal{A}$ supports a resonant pair of weights (λ,μ) then $\mathcal{A}$ supports a neighborly partition $\mathcal{P}$ and

$\exists$ two independent vectors in the kernel of the incidence matrix

$$\begin{array}{r|c|}
 & 1 \; - - - - \; n \\
X \not\subseteq \mathcal{B} \rightarrow & \\
\text{“multi-colored} \rightarrow & \\
\text{flats”} \rightarrow & \\
 &
\end{array}.$$

Ex:





$\mathcal{P} = 14 \mid 23 \mid 56$

☆ Find examples of arr.'s that support neighborly partitions.

Corolary

If $|X \cap \mathcal{B}| \geq |X| - 1 \Rightarrow X \subseteq \mathcal{B}$

If $|X| = 2$ then $X \cap \mathcal{B} \neq \emptyset \Rightarrow X \subseteq \mathcal{B}$.

$$\begin{array}{r c}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{array}{r} 126 \\ 135 \\ 245 \\ 346 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}
\end{array}$$

has nulli ≥ 2

$$\Rightarrow \quad \lambda = (1,0,0,1,-1,-1) = \text{red} - \text{green}$$

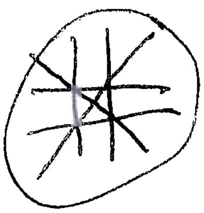
$$\mu = (0,1,1,0,-1,-1) = \text{blue} - \text{green}$$

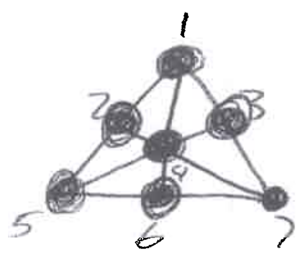
Conclusion: $(a_1+a_4-a_5-a_6) \wedge (a_2+a_3-a_5-a_6) = 0$

$$a_\lambda \wedge a_\mu = 0$$

$$\Rightarrow \mathbb{K}a_\lambda + \mathbb{K}a_\mu \subseteq R'(A)$$

Ex:





$P = 1|236|4|5|7$

6×7

$$\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 125 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 137 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 247 & & & & & & & \\ \vdots & & & & & & & \end{array}$$

$I =$ (6 rows)

$\text{nullity}_{\mathbb{C}}(I) = 1$

$\text{nullity}_{\mathbb{Z}_2}(I) = 3$

nullspace has basis: ________

over $\mathbb{Z}_2$, there is a vector in $R'_{\mathbb{Z}_2}(A)$

Applications of Characteristic Varieties

1. $1 \to K_\gamma \to G \xrightarrow{\gamma} \mathbb{Z}_N \to 1$

$$\longleftrightarrow \quad \begin{array}{ccc} \mathbb{Z}_N & \longrightarrow & X^\gamma \\ & & \downarrow \\ & & X \end{array}$$

cyclic regular N-fold cover of $X = X_G$

$\mathbb{K}$-field (sufficiently large w.r.t. $\mathbb{Z}_N$)

Then
$$\boxed{\dim_{\mathbb{K}} H^1(X^\gamma, \mathbb{K}) = n + \sum_{1 \neq k \mid N} \phi(k) \cdot \operatorname{depth}_{\mathbb{K}}(\gamma^{N/k})}$$

where

$$\operatorname{depth}_{\mathbb{K}}(t) = \max\{d \mid t \in V_d(G, \mathbb{K})\}$$

for $t \in \operatorname{Hom}(G, \mathbb{K}) \cong (\mathbb{K}^*)^n$

$$\begin{tikzcd} G \arrow[r, "\gamma"] \arrow[rrr, bend right, "\gamma^r"'] & \mathbb{Z}_N \arrow[r, hook] & \mathbb{K}^* \arrow[r, "x \mapsto x^r"] & \mathbb{K}^* \end{tikzcd}$$

$1 \longmapsto$ primitive N^{th} root of 1

2. Counting homomorphisms to finite grps

$$\delta_\Gamma(G) = \frac{|\operatorname{Epi}(G, \Gamma)|}{|\operatorname{Aut}(\Gamma)|}$$

G-finitely gen

Γ-finite

$(= \mathrm{GQuotients}(G,\Gamma)$ in $\mathcal{G}ap)$

Use characteristic varieties of G to compute these #s, in case Γ is metabelian, e.g. $\Gamma = S_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$

$\Gamma = A_4 = (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_3$

$$\delta_{S_3}(G) = \frac{1}{2}\sum_{d\geq 1} \left| \mathrm{Tors}_2\left(V_d(G,\mathbb{Z}_3) \setminus V_{d+1}(G,\mathbb{Z}_3)\right)\right| \cdot (3^d - 1)$$

$$\delta_{A_4}(G) = \frac{1}{6}\sum_{d\geq 1} \left| \mathrm{Tors}_3\left(V_d(G,\mathbb{F}_4) \setminus V_{d+1}(G,\mathbb{F}_4)\right)\right| \cdot (4^d - 1)$$

where $\mathrm{Tors}_p(V) = \{t \in V \mid t^p = 1,\ t \neq 1\}$ for $V \subset (\mathbb{k}^*)^n$

Milnor fibration

$\mathcal{A} = \{H_1, \dots, H_n\}$ arr. in $\mathbb{C}^\ell$, $H_i = \ker \alpha_i$ $\alpha_i : \mathbb{C}^\ell \to \mathbb{C}$ linear

$M = \mathbb{C}^\ell \setminus \bigcup H_i$ $\overline{M} = \mathbb{CP}^{\ell-1} \setminus \bigcup \overline{H}_i$

Fix weights $a = (a_1, \dots, a_n)$ $a_i \geq 1$ $\in \mathbb{Z}_{>0}$ set $N = \sum_{i=1}^{n} a_i$

$f : \mathbb{C}^\ell \longrightarrow \mathbb{C}$

$f = \alpha_1^{a_1} \cdots \alpha_n^{a_n}$ (homog. poly. of deg. N)

The restriction of f, $f : M \longrightarrow \mathbb{C}^*$ is the projection of a fiber bundle

(i.e. a submersion)

with • fiber $F = f^{-1}(1)$

• monodromy $h: F \to F$

$h(z_1, \dots, z_\ell) = (\xi z_1, \dots, \xi z_\ell)$

$\xi = e^{2\pi i/N}$

(variant: $f/|f| : M \to S^1$)

Basic Questions: **Note**: $h^N = Id$

• What is $H_*(F)$

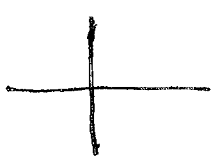
• what is $h_*: H_*(F) \to H_*(F)$

In Particular Question: What is $H_*(F)$ when $\mathcal{A}$ = braid arr. & weights $= 1$

$F = \{ \prod_{i<j} (z_i - z_j) = 1 \}$

Quick example

(i) $f: \mathbb{C}^2 \to \mathbb{C}$ $\quad f(x,y) = xy$

$\mathcal{A}$ 

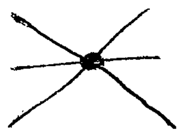
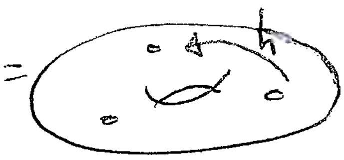
$$\begin{array}{ccc} F = \mathbb{C}^* & \to & M \\ & & \downarrow f \\ & & \mathbb{C}^* \end{array}$$

$H_1(F) = \mathbb{Z}$

$h_*: \mathbb{Z} \to \mathbb{Z}$ $\quad h_* = Id$

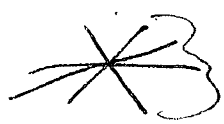
But $h \neq id$ and in fact h = Dehn twist

(ii) $f(x,y) = xy(x-y)$

$\mathcal{A}$  $F =$ 

Problem: compute $H_*(F)$ (Hint: $H_1(F) = \mathbb{Z}^4$)
and $h_* : H_*(F) \to H_*(F)$

More generally

$\mathcal{A}$  n $\quad H_1(F) = \mathbb{Z}^{(n-1)^2}$

Recall Hopf fibration

$$
\begin{tikzcd}
\mathbb{Z}_N & \mathbb{C}^* \arrow[d] & \mathbb{C}^* \arrow[d] \\
F \arrow[r, phantom, "\subset"] \arrow[dr, "\pi_F"'] & M \arrow[r, phantom, "\subset"] \arrow[d, "\pi"] & \mathbb{C}^2 \setminus \{0\} \arrow[d, "\pi"] \\
& \bar{M} \arrow[r, phantom, "\subset"] \arrow[u, bend right, "s"'] & \mathbb{CP}^{2-1}
\end{tikzcd}
$$

Putting things together

$$
\begin{tikzcd}
& & \mathbb{Z}_n \arrow[d] & \\
& & F \arrow[d] \arrow[dr, "\pi_F"] & \\
\mathbb{Z}_N \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[dr, "z \mapsto z^N"'] & M \arrow[r] \arrow[d] & \bar{M} \arrow[l, bend left, "s"] \arrow[dl, "\varphi"] \\
& & \mathbb{C}^* &
\end{tikzcd}
$$

ie.
$$
\begin{tikzcd}
\mathbb{Z}_N \arrow[r] & F \arrow[d] \\
& \bar{M}
\end{tikzcd}
$$
is pullback of cannonical N-fold cover
$$
\begin{tikzcd}
\mathbb{Z}_N \arrow[r] & \mathbb{C}^* \arrow[d] \\
& \mathbb{C}^*
\end{tikzcd}
$$

Passing to homotopy

$$\begin{tikzcd}
 & & & & 1 \arrow[d] & & 1 \arrow[dll] & & \\
 & & & & \pi_1(F) \arrow[d] \arrow[dll] & & & 1 \arrow[dl] & \\
 & 1 \arrow[r] & \pi_1(\bar{M}) \arrow[rr, bend right, "s_*"'] \arrow[dl] & & \pi_1(M) \arrow[ll, bend right, "\pi_*"'] \arrow[rr] \arrow[d] & & \mathbb{Z} \arrow[rr] \arrow[ll, bend right] \arrow[dll, "\cdot N"] & & 1 \\
 & \mathbb{Z}_N \arrow[dl] & & & \mathbb{Z} \arrow[d] \arrow[dl] & & & & \\
1 & & & \mathbb{Z}_N \arrow[dl] & 1 & & & & \\
 & & 1 & & & & & &
\end{tikzcd}$$

Covering $\begin{array}{c} F \\ \downarrow \\ \bar{M} \end{array}$ is classified by

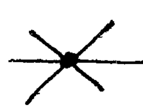
$$\gamma : \pi_1(\bar{M}) \to \mathbb{Z}_N$$

$$x_i \longmapsto a_i$$

From result before, we get

$$\boxed{\dim_{\mathbb{K}} H_1(F, \mathbb{K}) = n-1 + \sum_{1 \neq k | N} \phi(k) \cdot \operatorname{depth}_{\mathbb{K}} \left(\gamma_a^{N} \right.}$$

Example 1

$\mathcal{A} =$  $f(x,y) = xy(x-y)$

$t_1 = w$, $t_2 = w$, $(t_1 t_2)^{-1} = (w^2)^{-1}$

$V_1 = \{ t_1 t_2 t_3 - 1 = 0 \} = \{ (t_1, t_2, (t_1 t_2)^{-1}) \}$

$V_2 = \{1\}$

$N=3$

$$b_1(F) = 2 + \underset{\substack{\shortparallel \\ 2}}{\phi(3)} \cdot \underset{\substack{\shortparallel \\ 1}}{\mathrm{depth}(\gamma^1)} \qquad \gamma = \gamma_{(1,1)} = (\omega, \omega, \omega^{-1})$$

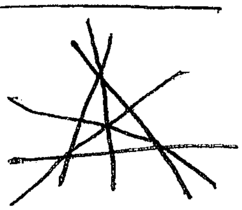
$\omega = e^{2\pi i/3}$

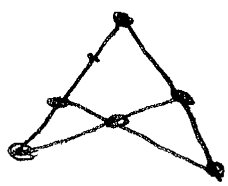
$= 4$

$$\delta_{S_3}(G) = \frac{1}{2}\left(\underset{2^2-1}{\mathrm{Tors}_2 V_1(G,\mathbb{Z}_3) \setminus V_2(G,\mathbb{Z}_3)}\right)(3-1) = 3$$

$$\delta_{A_4}(G) = \frac{1}{6}\left(\underset{3^2-1}{\mathrm{Tors}_3 (V_1(G,\mathbb{Z}_3) \setminus V_2(G,\mathbb{Z}_3))}\right)(4-1) = 4$$

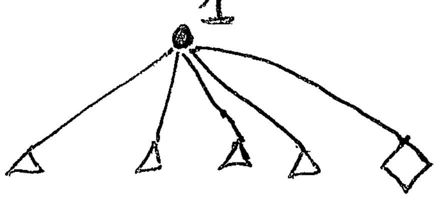
Example 2 (Braid arr.)

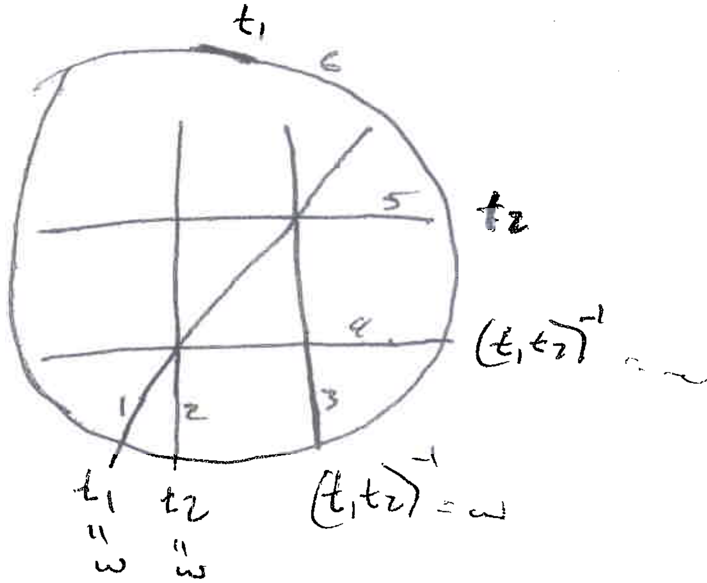




$f = x_1 x_2 x_3 (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$

Char. Varieties





$C = \{t_1 = t_6,\ t_3 = t_4,\ t_2 = t_5,\ t_1 \cdots t_6 = 1\}$

$F = f^{-1}(1) \qquad \gamma = \gamma_a : \pi_1(\bar{M}) \to \mathbb{Z}_6 \subset k^*$

$\gamma = (\zeta, \dots, \zeta) \qquad \zeta = e^{2\pi i/6}$

$$b_1(F) = 5 + \underset{1}{\underset{\shortparallel}{\phi(2)}}\,\underset{0}{\underset{\shortparallel}{\cancel{\text{depth}}(\gamma^3)}} + \underset{2}{\underset{\shortparallel}{\phi(3)}}\,\underset{1}{\underset{\shortparallel}{\cancel{\text{depth}}(\gamma^2)}}$$

$$+ \underset{2}{\underset{\shortparallel}{\phi(6)}}\,\underset{0}{\underset{\shortparallel}{\cancel{\text{depth}}(\gamma)}}$$

$$\zeta = \omega$$

$$= 7$$

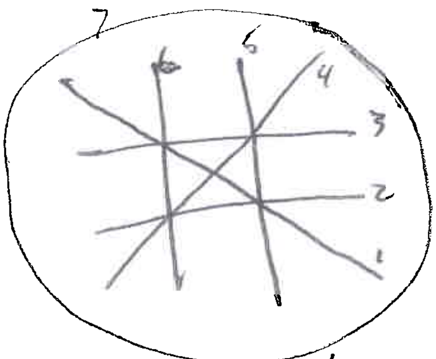
$$\delta_{S_3} = 5 \cdot 3 = 15$$

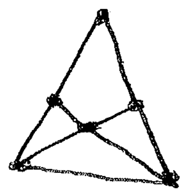
in fact $H_1(F) = \mathbb{Z}^7$

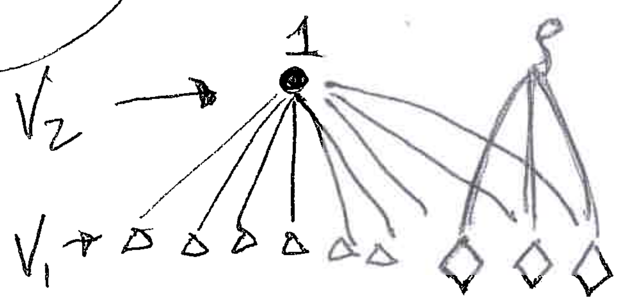
$$\delta_{A_4} = 5 \cdot 4 = 20$$

Ex: 3 (non-fano)

$$f = xyz(x-y)(x-z)(y-z)(x+y-z)$$







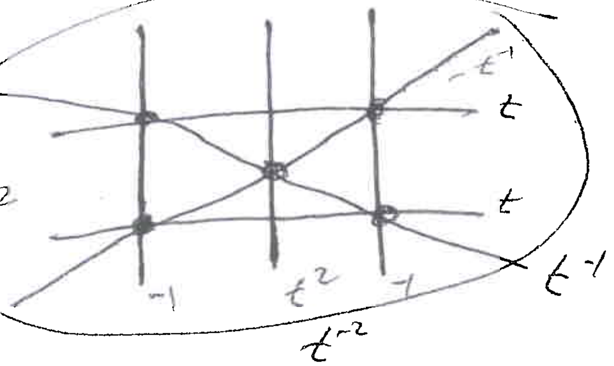
$$\rho = (1, -1, -1, 1)$$

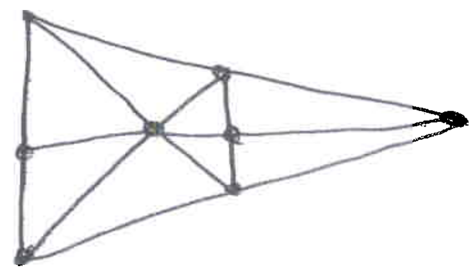
$$b_1(F) = 6 + \underset{6}{\underset{\shortparallel}{\phi(7)}}\,\underset{0}{\underset{\shortparallel}{\text{depth}(\gamma)}} = 6$$

$$\delta_{S_3}(G) = \frac{1}{2}\overbrace{9}^{\#\text{ of components}} \cdot \underset{27}{\underset{\shortparallel}{(3-1)(2^2-1)}} \; -3 + \frac{1}{2}(3^2-1) =$$

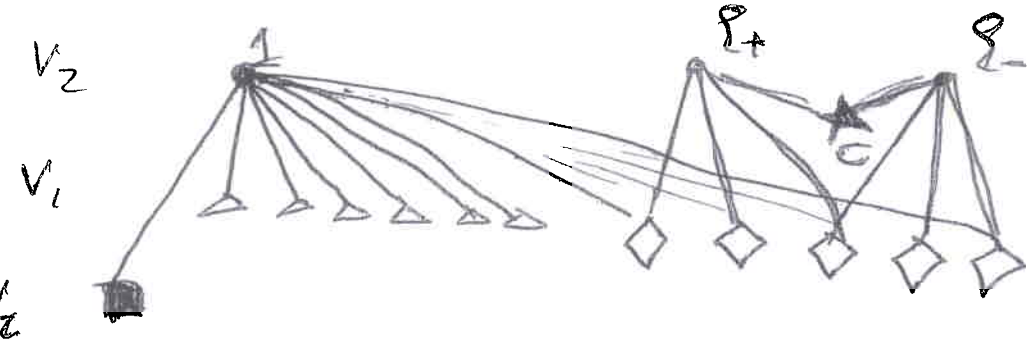
$$\delta_{A_4}(G) = 9 \cdot 4 = 36$$

Example 4 (Deleted B_3)





$$Q = x_1 x_2 (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2)$$



$C \equiv$ 1-dim. torus which does not go thru 1

$C = \{(t^2, t^{-2}, -1, -1, t^{-1}, -t^{-1}, t, t)\} = \rho \cdot T$ $\quad \rho^2 = 1 \ \rho \neq 1$

<u>Milnor fibration with</u> $a = (2,1,3,3,2,2,1,1)$

$N = 15$ decone w.r.t. x_2

$F \xrightarrow{\mathbb{Z}_{15}} M$ given by $\gamma_a = (\zeta^2, \zeta, \zeta^3, \zeta^3, \zeta^2, \zeta^2, \zeta, \zeta)$

where $\zeta = e^{2\pi i/15}$

$b_1(F, \mathbb{K}) = 7 + \phi(3)\, d(\gamma^5) + \phi(5)\, d(\gamma^3) + \phi(15)\, d(\gamma)$

if char $\mathbb{K} \neq 2$ $\quad b_1(F, \mathbb{K}) = 0 + 7$ $\quad \gamma^5 = (\omega^2, 1, 1, \omega^2, \omega^2, \omega, \omega) \notin C$

" " " $= 2$ $\quad b_1(F, \mathbb{K}) = 9$

in fact $H_1(F) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

(49)

$$\delta_{S_3}(G) = \frac{1}{2}(2^3-1)(3^2-1) + 11 \cdot \frac{1}{2}(2^2-1)(3-1)$$
$$-6 + 2 \cdot \frac{1}{2}(3^2-1) = 63$$

$$\delta_{A_4}(G) = \frac{1}{6}(3^3-1)(4^3-1) + 11 \cdot \frac{1}{6}(3^2-1)(4-1)$$
$$+ 1 \cdot \frac{1}{6}(3-1)(4-1)$$
$$= 110$$

§1) Combinatorial Decomposition of $R^1(G,R)$

$\mathcal{A}$ = central arr. w/ underlying matroid G

R = field

<u>Def</u>: $R^1(G,R) = \{\lambda \in R^n \mid \dim_R H^1(A, a_\lambda) \geq 1\}$

(R not a field

$$R^1(A,R) = \{\lambda \in R^n \mid \exists\, \mu \in R^n \text{ s.t. } \lambda \not\parallel \mu \text{ and } a_\lambda \wedge a_\mu = 0\}$$

where $\lambda \parallel \mu \iff$ all 2×2 minors of $[\lambda \mid \mu]$ vanish

<u>Thm</u>: $R^1(A,R) = \bigcup\limits_{\Gamma \in NP(G,R)} V^1(\Gamma, R)$

$NP(G)$ = set of neighborly partitions of submatroids of G

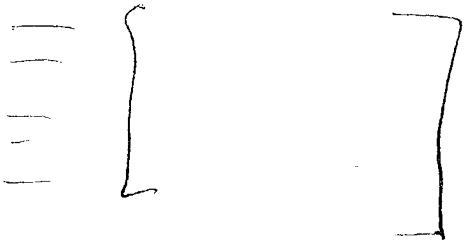
$\Gamma \in NP(G)$, a neighborly partition of $G' \leq G$

(wlog G' has ground set $[m]$)

define $K(\Gamma, R) = \{\lambda \in R^m \mid \lambda_X = 0 \ \forall$ rank two flat X not contained in a block of $\Gamma\}$

= kernel of pt-line incidence matrix

multi-colored slots of G'



$$NP(G,R) = \{\Gamma \in NP(G) \mid \dim_R K(\Gamma,R) \geq 2\}$$

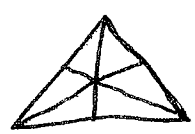
$$V'(\Gamma,R) = \{\lambda \in K(\Gamma,R) \mid \exists\, \mu \in K(\Gamma,R),\ \mu \nparallel \lambda \text{ s.t. } \lambda^S \parallel \mu^S \ \forall \text{ blocks } S \text{ of } \Gamma\}$$

restrict to S

$\forall\, i \sim j$

$$\det \begin{bmatrix} \lambda_i & \mu_i \\ \lambda_j & \mu_j \end{bmatrix} = 0$$

Ex:



has nullity$_{\mathbb{C}} = 1$ of the incidence matrix

§2) Line structure of $\overline{V'(\Gamma,R)}$

Note: $\lambda \in V'(\Gamma,R)$, $c \in R - \{0\} \Rightarrow c\lambda \in V'(\Gamma,R)$

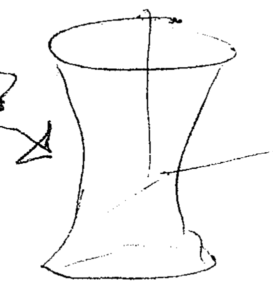
$\overline{V'(\Gamma,R)}$ = proj. image of $V'(\Gamma,R)$

Note: $\lambda \in V'(\Gamma,R) \Rightarrow \exists\, \mu \in V'(\Gamma,R)$ s.t.

$$R\lambda + R\mu \subseteq V'(\Gamma,R)$$

$\overline{R\lambda + R\mu}$ is a proj. line in $\overline{V'(\Gamma,R)}$

i.e. $\overline{V'(\Gamma,\mathbb{R})}$ is "ruled by lines"



ex

call $\lambda * \mu := \overline{\mathbb{R}\lambda + \mathbb{R}\mu}$

Note in addition:

If S is a block of Γ, then the line $\lambda * \mu$ intersects the subspace

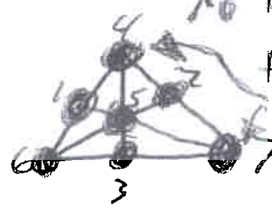
$$\overline{D}_S = \overline{\{\eta \in K(\Gamma,\mathbb{R}) \mid \eta^S = 0 \quad \eta_i = 0 \ \forall i \in S\}}$$

pf: By hypothesis $\lambda^S \,\|\, \eta^S$ Then $\exists\, a,b \in \mathbb{R}$ s.t. $a\lambda^S + b\mu^S = 0$

Set $\eta = a\lambda + b\mu$ then $\bar{\eta} \in (\lambda * \mu) \cap \overline{D_S}$

<u>Thm</u> $\overline{V'(\Gamma,\mathbb{R})}$ is the union of the lines in $\overline{K(\Gamma,\mathbb{R})}$ which meet $\overline{D}_S$ for every block S of Γ

<u>Ex:</u>



λ_0 is non-zero on these coordinates

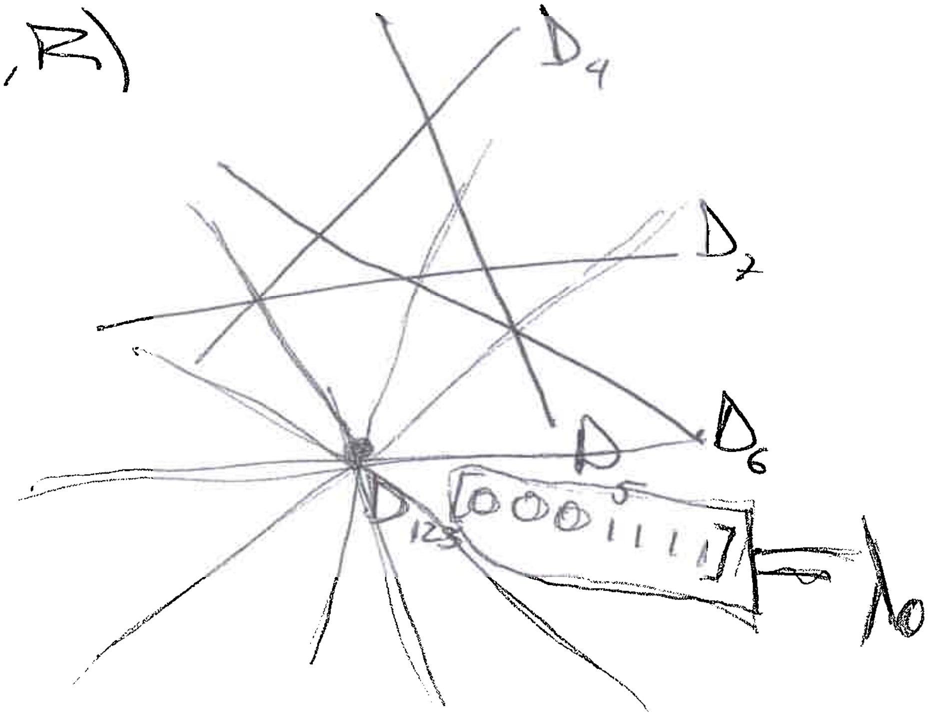
$\Gamma = 123/4/5/6/7$

$K(\Gamma, \overline{\mathbb{Z}_2}) \cong \mathbb{R}^3$ so $\overline{K(\Gamma,\mathbb{R})} \cong \mathbb{P}^2$

$\| \mathbb{R}$

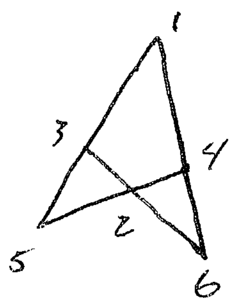
alg. closure

$\overline{K(\Gamma, R)}$



$$\overline{V'(\Gamma, R)} = \overline{K(\Gamma, R)} \cong \mathbb{P}^2$$

Ex Braid arr.



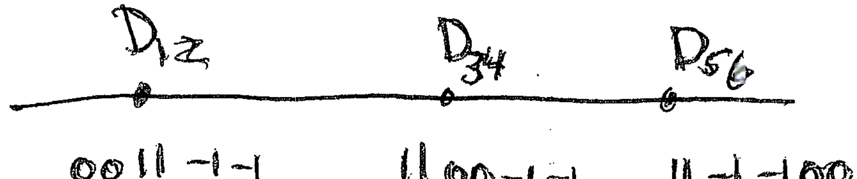
incidence matrix is

$$\begin{array}{cc} & 6 \\ 4 & \left[\begin{array}{c} \quad\quad\quad\quad \end{array}\right] \end{array}$$

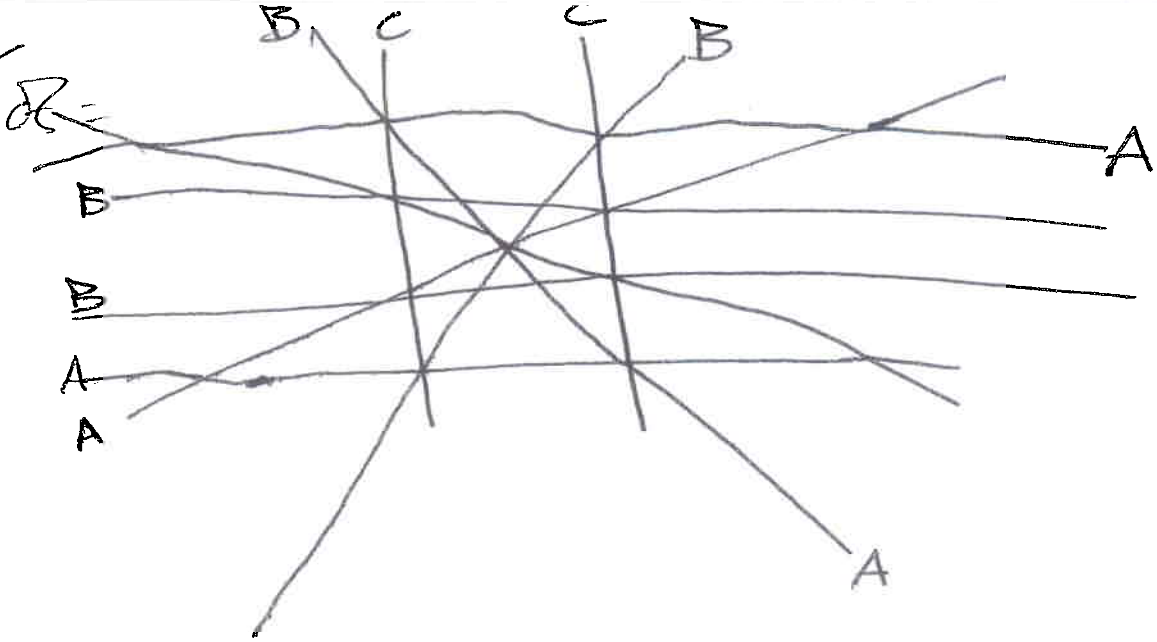
$\Gamma = 12|34|56$

So, $K(\Gamma, R) \cong R^2$ for any R,

$$\Rightarrow \overline{K(\Gamma, R)} = \mathbb{P}$$
$$\|$$
$$\overline{V'(\Gamma, R)}$$



Ex/ $\mathcal{A} =$

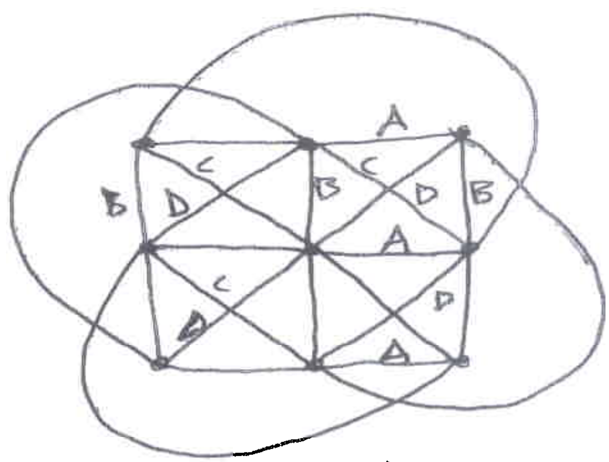


$R = \overline{\mathbb{Z}_2}$ $\quad \overline{K(\Gamma, R)} = \mathbb{P}^3$

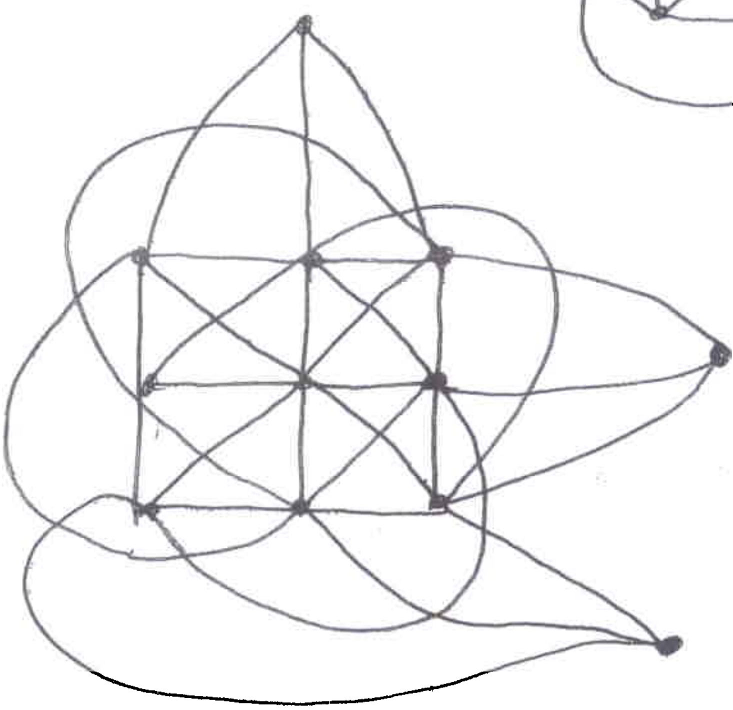
$\overline{V'(\Gamma, R)} \neq \overline{K(\Gamma, R)}$

Ex/ $\mathcal{A} =$ Hessian

12 lines in $\mathbb{P}^2$



$G =$



4 blocks of size 3

$\dim K(\Gamma, R) = 3$ if $R = \mathbb{C}$ and $\overline{V'(\Gamma, R)} = \overline{K(\Gamma, R)}$

If $R = \overline{\mathbb{Z}_3}$ then $\dim K(\Gamma, R) = 6$

$\overline{K(\Gamma, R)} \cong \mathbb{P}^5$ 4-directrices

are $\mathbb{P}^2$'s $\subseteq \mathbb{P}^4$

$\overline{V'(\Gamma, R)}$ has degree 3 is

irreducible and dim. = 3.

LCS Formula (Falk)

$\mathcal{A}$ central arr. $G = \pi_1(M)$

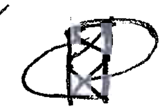
$G = G_1 \geq G_2 \geq \cdots \geq G_n \geq \cdots$ L.C.S.

$G_i = [G_{i-1}, G]$

$\varphi_i(\mathcal{A}) = \operatorname{rank} G_i / G_{i+1}$

G_i / G_{i+1} abelian, f.g. not necessarily torsion free

$P_M(t) = \operatorname{Poin}(\mathcal{A}, t)$

()

LCS formula

$$\prod_{k \geq 1} (1 - t^k)^{\varphi_k(\mathcal{A})} = P_M(-t)$$

First ex:  H $\mathcal{A} = \{H\}$ $G = \mathbb{Z}$

$\varphi_1 = 1$ $\varphi_2 = 0 = \varphi_k$ $k \geq 2$

$(1 - t) = 1 - t$

First Example

Kohno: $\mathcal{A} = \mathcal{A}_\ell$ – Braid arr.

$$\Rightarrow \quad \prod_{k \geq 1} (1-t^k)^{\phi_k} = (1-t)(1-2t)(1-3t)\cdots(1-(\ell-1)t)$$

Pf: rational homotopy theory / holonomy Lie alg.

$\mathcal{A}_\ell$ is supersolvable / fiber-type w/ exponents $1, 2, \ldots, \ell-1$

$$\begin{array}{ccccccc} M(\mathcal{A}_\ell) & \longrightarrow & M(\mathcal{A}_{\ell-1}) & \longrightarrow & M(\mathcal{A}_{\ell-2}) & \to \cdots \to & M(\mathcal{A}) = \mathbb{C}^* \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathbb{C}\setminus(\ell-1 \text{ pts}) & \longrightarrow & \mathbb{C}\setminus(\ell-2 \text{ pts}) & \longrightarrow & \cdots & & \end{array}$$

$$\pi_1(\mathbb{C}\setminus(d \text{ pts})) \cong F_d$$

Witt's formula

$$\prod_{k \geq 1} (1-t^k)^{\phi_k(F_d)} = 1 - dt$$

$$\phi_k(F_d) = \sum_{r \mid k} \frac{1}{k} \mu\left(\frac{k}{r}\right) d^r$$

$$\begin{tikzcd} \mathbb{C}\setminus(d\text{ pts}) \arrow[r] & M(\mathcal{A}) \arrow[d, "\text{projection}", "\text{fiber bundle}"'] \\ & M(\mathcal{B}) \end{tikzcd}$$

fiber type
$\Downarrow$
$\big(M(\mathcal{B})$ is aspherical$\big)$

long exact htpy seq.

$$(\mathcal{B})) \to \pi_1(\mathbb{C}\text{-}d\text{pts}) \to \pi_1(M(\mathcal{A})) \to \pi_1(M(\mathcal{B})) \to 1$$

$\|$
1

So we have

$$1 \to \overset{A}{\overset{\|}{F_d}} \to \overset{B}{\overset{\|}{\pi_1(M(\mathcal{A}))}} \underset{\sigma}{\rightleftarrows} \overset{C}{\overset{\|}{\pi_1(M(\mathcal{B}))}} \to 1$$

Ingredients: (i) $\exists$ a section

(ii) action of $\pi_1(M(\mathcal{B}))$ on F_d is trivial on $F_d/[F_d, F_d]$

$$1 \to A \to B \underset{\sigma}{\rightleftarrows} C \to 1$$

$\overset{\text{hard work}}{\Longrightarrow}$ $1 \to A_n \to B_n \rightleftarrows C_n \to 1$

$$\Longrightarrow 1 \to A_n/A_{n+1} \to B_n/B_{n+1} \to C_n/C_{n+1} \to 1$$

from free gp

Conclusion (by induction)

B_n/B_{n+1} is free abelian and

$$\operatorname{rk}\left(B_n/B_{n+1}\right) = \operatorname{rk}\left(C_n/C_{n+1}\right) + \operatorname{rk}\left(A_n/A_{n+1}\right)$$

for $\mathcal{A}$ fiber type w/ ~~exponents~~ fibers $\mathbb{C} - d_i$ pts $_{i=1,\dots,\ell}$

$\Longrightarrow$ ① $P_M(t) = \prod_{i=1}^{\ell}(1+d_i t)$

② $\varphi_k(G) = \sum_{i=1}^{\ell} \varphi_k(F_{d_i})$ $\quad \prod_{k\geq 1}(1-t^k)^{\varphi_k(G)}$

③ G_n/G_{n+1} is free Abelian

$$\prod_{i=1}^{\ell}\prod_{k\geq 1}(1-t^k)^{\varphi_k(F_{d_i})} = \prod_{k\geq 1}(1-t^k)^{\sum\varphi_k(F_{d_i})}$$

$$\overset{\text{Witt}}{\|}$$

$$\prod_{i=1}^{\ell}(1-d_i t) = P_M(-t)$$

<u>Thm</u> If $\mathcal{A}$ is fiber type then this LCS formula holds.

of fiber type

<u>Cor.</u> $\bigcap_{n\geq 1} G_n = 1$

i.e. G is residually nilpotent.

<u>Suciu</u>

<u>LCS:</u> $G = G_1 \supset G_2 \supset \cdots \supset G_r \supset \cdots$

$\operatorname{rank} G_r = \varphi_r$

LCS Quotients G_r/G_{r+1}

Nilpotent quotients G/G_r

$G/G_2 = G^{ab} = \mathbb{Z}^n$

$\to G_2/G_3 \to G/G_3 \longrightarrow G/G_2 \longrightarrow 1$

$\vdots$

$1 \to G_r/G_{r+1} \to G/G_{r+1} \to G/G_r \to 1$

<u>ex:</u> $G = F_2 = \langle x, y \rangle$

$\to G_2/G_3 \to G/G_3 \xrightarrow{ab} G/G_2 \longrightarrow 1$

$G_2/G_3 = [x,y]$, $\quad G/G_2 = \mathbb{Z}^2$ (generated by x, y)

$$G/G_3 = \langle x, y \mid [x,[x,y]] = [y,[x,y]] = 1 \rangle$$

$$\text{Heisenberg grp} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{Z} \right\}$$

<u>Thm:</u> $G = \pi_1(M)$

$K = \pi_1(N)$

$$G/G_3 \cong K/K_3 \iff H^{\leq 2}(M) \cong H^{\leq 2}(N)$$

(in particular G/G_3 is combinatorially determined)

<u>Warning</u> G/G_4 is <u>not</u> comb. determined.

Let
$$\nu_{p,d}(G) := \#\left\{ K \triangleleft G/G_3 \,\middle|\, \begin{array}{l} [G/G_3 : K] = p \\ \dim_{\mathbb{Z}_p} H^1(K, \mathbb{Z}_p) = n+d \end{array} \right\}$$

p is prime, $d \geq 0$

<u>Thm:</u> $$\nu_{p,d}(G) = \frac{1}{p-1} \left| R_d(G, \mathbb{Z}_p) \setminus R_{d+1}(G, \mathbb{Z}_p) \right|$$

Associated graded Lie alg.

$$gr(G) = \bigoplus_{r=1}^{\infty} \underbrace{G_r / G_{r+1}}_{gr_r(G)}$$

$$[\ ,\ ] : gr_k(G) \times gr_s(G) \longrightarrow gr_{k+s}(G)$$

given by commutator $[x,y] = xyx^{-1}y^{-1}$

$[x,y] = -[y,x]$ and Jacobi identity

Ex: $G = F_n$ free grp

$gr(G) = L_n$ free Lie alg.

$U(L_n) = T_n$ = tensor alg.

univer. enveloping alg. $\phi_r = \operatorname{rank} gr_r(G)$

PBW Thm

$$\prod_{r=1}^{\infty} (1-t^r)^{-\phi_r} = \mathrm{Hilb}(U(gr(G)))$$

Ex: $G = F_n$ $\displaystyle\prod_{r=1}^{\infty} (1-t^r)^{-\phi_r(F_n)} = \mathrm{Hilb}(T_n) = \frac{1}{1-nt}$

Chen Lie alg (~1950)

$\underline{gr}(G/G'')$ $\qquad \theta_r = \mathrm{rank}\ \underline{gr}_r(G/G'')$

this is a quotient of $\underline{gr}(G)$, so

$\varphi_r \geq \theta_r$ also $\varphi_i = \theta_i$ for $i=1,2,3$

Thm: (Massey)

$$\theta_r = \dim_{\mathbb{C}} gr_{r-2} B \otimes \mathbb{C}$$

$$(B = G'/G'' \text{ Alex inv.})$$

filtered by I^r

$$\underline{gr}_r B = I^r B / I^{r+1} B$$

in particular

$$\varphi_2 = \theta_2 = \dim B / I B$$

$$\varphi_3 = \theta_3 = \dim IB / I^2 B$$

Theorem (PSO)

$$\theta_r = \dim gr_{r-2} B^{lin} \otimes \mathbb{C}$$

combinatorially determined

Application

Chen • $\theta_r(F_n) = (r-1)\binom{n+r-2}{r}$ $\quad r \geq 2$

C.S. 94 $\quad \theta_r(P_n) = (r-1)\binom{n+1}{4}$ for $r \geq 2$

$\quad \#$

$\quad \theta_r(F_{n-1} \times \cdots \times F_1)$ for $n \geq 4$

Digression

Recall

$$R_d(G_\chi) = V(\mathrm{ann}\, B^{\mathrm{lin}})$$

<u>Question:</u> Can we write θ_r (and maybe ϕ_r) in terms of $R_d(\mathcal{A})$?

Resonance LCS Conjectures (2000)

$R_1 = R^1_1 = \bigcup_{i=1}^{s} L_i \quad , L_i$ linear sbspce of $\mathbb{C}^n$

$h_r := \#\{L_i \mid \dim L_i = r\}$ $\quad$ [= for non-resonant arr's] [= for graph arr's] [$\geq$ true (SS 04)]

Then :

• $\theta_k = \sum_{r \geq 2} h_r \cdot \theta_k(F_r)$ for $k \gg 0$.

• $\phi_k = \sum_{r \geq 2} h_r \phi_k(F_r)$ if $\theta_4 = \phi_4$ for $k \gg 0$

Thm: If $\Phi_3 = \sum_{x \in L_2(\partial\tau)} \Phi_3(F_{\mu(x)})$ then

$$\prod_{r=1}^{\infty} (1-t^r)^{\Phi_r(\partial\tau)} = (1-t)^{|\partial\tau| - \sum_{x \in L_2(\partial\tau)} \mu(x)} \prod_{x \in L_2(\partial\tau)} (1-\mu(x)t)$$

THE END