Lecture 16 Falk

§1) Combinatorial Decomposition of $R^1(G,\mathbb{k})$

$\mathcal{A}$ = central arr. w/ underlying matroid G

$\mathbb{k}$ = field

Def: $R^1(G,\mathbb{k}) = \{\lambda \in \mathbb{k}^n \mid \dim_{\mathbb{k}} H^1(A, a_\lambda) \geq 1\}$

(R not a field

$$R^1(A,R) = \{\lambda \in R^n \mid \exists\, \mu \in R^n \text{ s.t. } \lambda \not\parallel \mu \text{ and } a_\lambda a_\mu = 0\}$$

where $\lambda \| \mu \iff$ all 2×2 minors of $[\lambda \mid \mu]$ vanish

Thm: $R^1(A,\mathbb{k}) = \bigcup_{\Gamma \in NP(G,\mathbb{k})} V^1(\Gamma, \mathbb{k})$

$NP(G)$ = set of neighborly partitions of submatroids of G

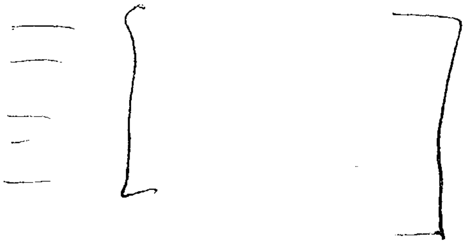
$\Gamma \in NP(G)$, a neighborly partition of $G' \leq G$

(wlog G' has ground set $[m]$)

define $K(\Gamma,\mathbb{k}) = \{\lambda \in \mathbb{k}^m \mid \lambda_X = 0 \ \forall$ rank two flat X not contained in a block of $\Gamma\}$

= kernel of pt-line incidence matrix

multi-colored slots of G'



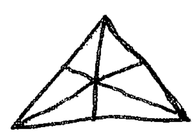
$$NP(G,R) = \{\Gamma \in NP(G) \mid \dim_R K(\Gamma,R) \geq 2\}$$

$$V'(\Gamma,R) = \{\lambda \in K(\Gamma,R) \mid \exists\, \mu \in K(\Gamma,R),\ \mu \nparallel \lambda \text{ s.t. } \lambda^S \parallel \mu^S \ \forall \text{ blocks } S \text{ of } \Gamma\}$$

restrict to S

$\det \begin{bmatrix} \lambda_i & \mu_i \\ \lambda_j & \mu_j \end{bmatrix} = 0 \quad \forall i,j$

Ex:



has nullity$_{\mathbb{C}} = 1$ of the incidence matrix

S2) Line structure of $\overline{V'(\Gamma,R)}$

Note: $\lambda \in V'(\Gamma,R),\ c \in R - \{0\} \Rightarrow c\lambda \in V'(\Gamma,R)$

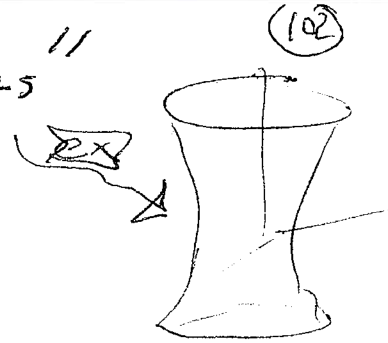
$\overline{V'(\Gamma,R)}$ = proj. image of $V'(\Gamma,R)$

Note: $\lambda \in V'(\Gamma,R) \Rightarrow \exists\, \mu \in V'(\Gamma,R)$ s.t.

$R\lambda + R\mu \subseteq V'(\Gamma,R)$

$\overline{R\lambda + R\mu}$ is a proj. line in $\overline{V'(\Gamma,R)}$

i.e. $\overline{V'(\Gamma,\mathbb{R})}$ is "ruled by lines"



call $\lambda * \mu := \overline{\mathbb{R}\lambda + \mathbb{R}\mu}$

Note in addition:

If S is a block of Γ, then the line $\lambda * \mu$ intersects the subspace

$$\overline{D}_S = \overline{\{\eta \in K(\Gamma,\mathbb{R}) \mid \eta^S = 0 \quad \eta_i = 0 \ \forall i \in S\}}$$

pf: By hypothesis $\lambda^S \| \mu^S$ Then

$\exists\ a, b \in \mathbb{R}$ s.t. $a\lambda^S + b\mu^S = 0$

Set $\eta = a\lambda + b\mu$ then $\bar{\eta} \in (\lambda * \mu) \cap \overline{D}_S$

Thm $\overline{V'(\Gamma,\mathbb{R})}$ is the union of the lines in $\overline{K(\Gamma,\mathbb{R})}$ which meet $\overline{D}_S$ for every block S of Γ

Ex:

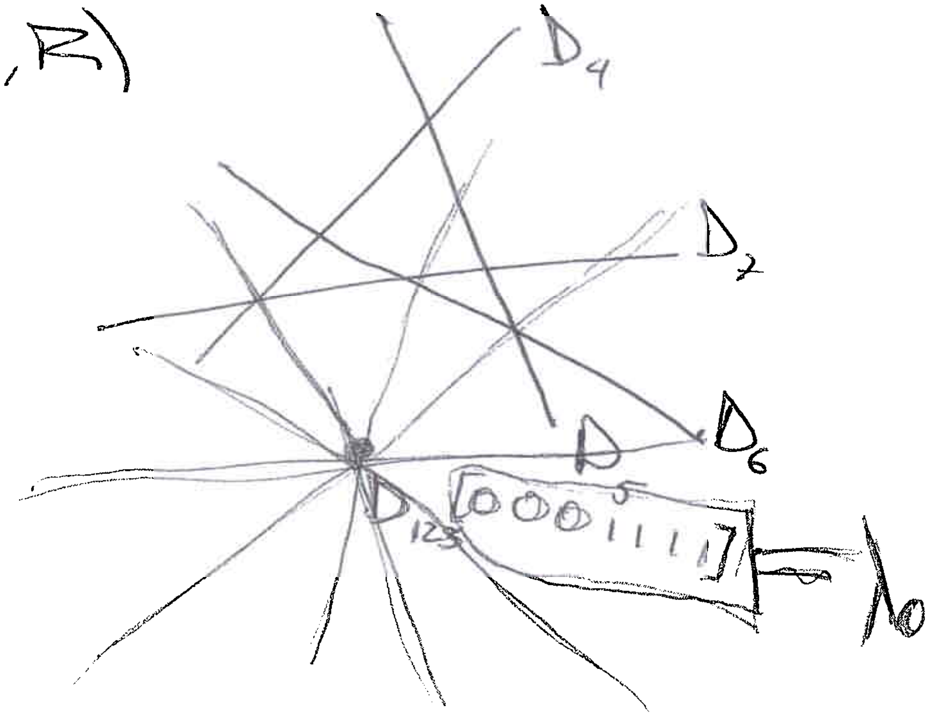
$\Gamma = 123/4/5/6/7$

$K(\Gamma,\mathbb{Z}_2) \cong \mathbb{R}^3$ so $\overline{K(\Gamma,\mathbb{R})} \cong \mathbb{P}^2$

$\| \mathbb{R}$

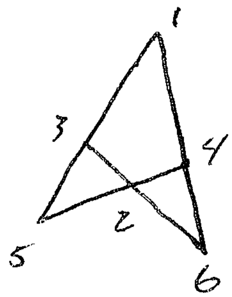
alg. closure

$\overline{K(\Gamma,R)}$



$$\overline{V'(\Gamma,R)} = \overline{K(\Gamma,R)} \cong \mathbb{P}^2$$

Ex Braid arr.



incidence matrix is

$$\begin{array}{cc} & 6 \\ 4 & \begin{bmatrix} \quad & \quad \end{bmatrix} \end{array}$$

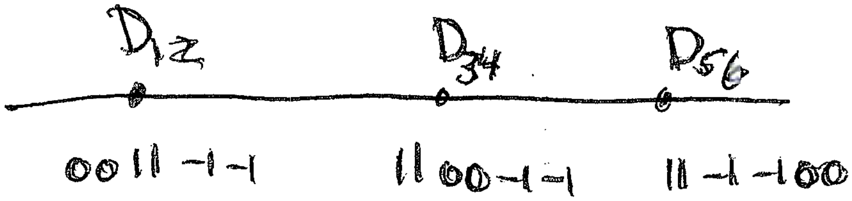
$\Gamma = 12|34|56$

So, $K(\Gamma,R) \cong R^2$ for any R,

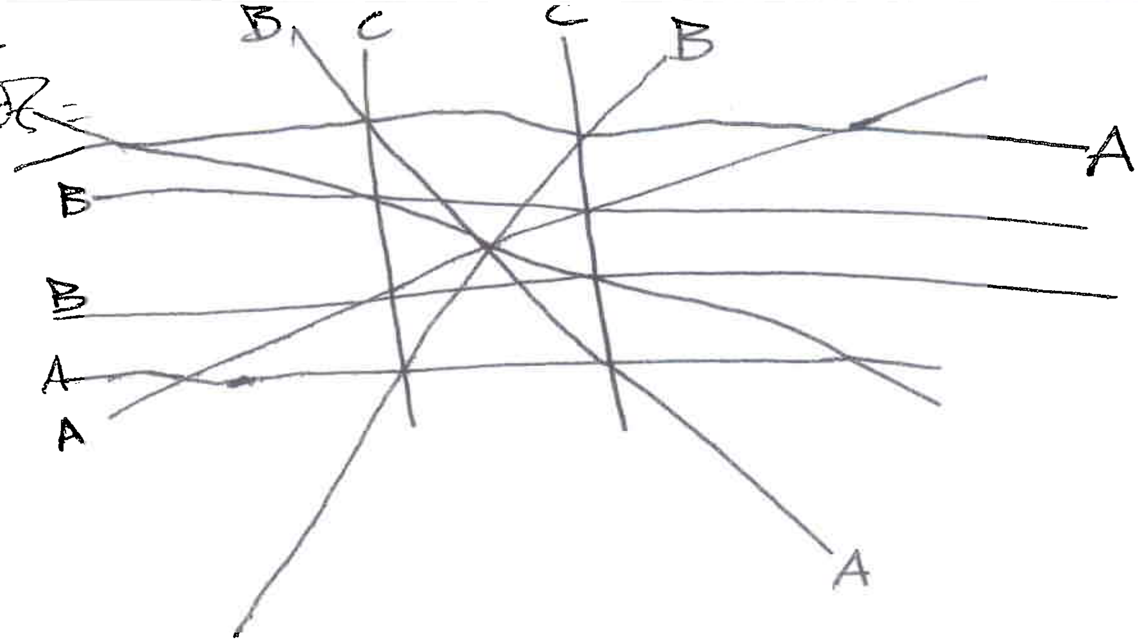
$$\Rightarrow \overline{K(\Gamma,R)} = \mathbb{P}$$

$$\|$$

$$\overline{V'(\Gamma,R)}$$



Ex/ $\mathcal{A} =$

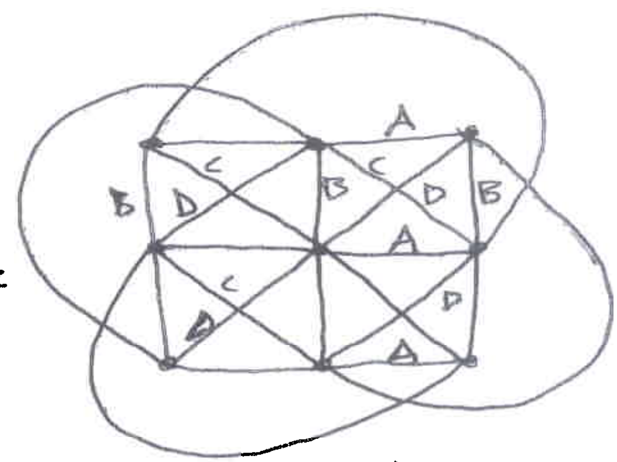


$R = \overline{\mathbb{Z}_2}$ $\overline{k(\Gamma, R)} = \mathbb{P}^3$

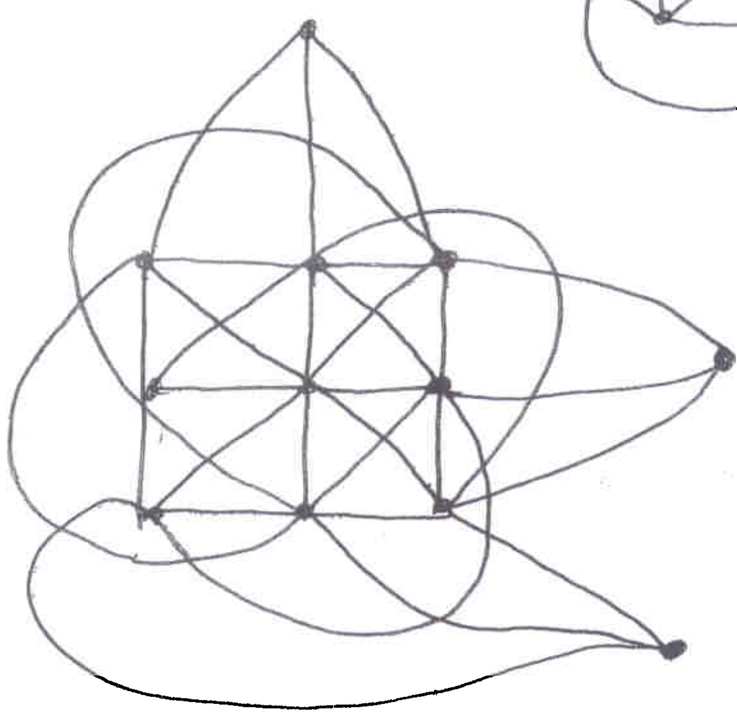
$\overline{V'(\Gamma, R)} \neq \overline{k(\Gamma, R)}$

Ex/ $\mathcal{A} =$ Hession

12 lines in $\mathbb{P}^2$



$G =$



4 blocks of size 3

dim $k(\Gamma, R) = 3$ if $R = \mathbb{C}$ and

$\overline{V'(\Gamma, R)} = \overline{k(\Gamma, R)}$