Lecture 14 — Falk

I. Genericity Conditions

(ESV + STV)

$\mathcal{A}$: essential affine arr. in $\mathbb{C}^\ell$

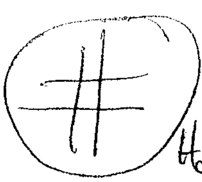
$\lambda \in \mathbb{C}^n \longleftrightarrow t = \exp(2\pi i \lambda) \in (\mathbb{C}^*)^n$

$a_\lambda = \sum_{i=1}^{n} \lambda_i a_i$ $\qquad \mathcal{L}_t$ = rank 1 local system on M determined by t

$X \in L \longrightarrow \lambda_X = \sum_{H_i \supseteq X} \lambda_i$

(ESV + STV) If $\lambda_X \notin \mathbb{Z}_{>0}$ for every irreducible $X \in L(\mathcal{A}_\infty)$, $X \neq 0$ (i.e. $X \in L(c\mathcal{A})$, $X \neq \mathbb{1}_\ell$) then $H^*(M, \mathcal{L}_t) = H^*(A, a_\lambda)$

$\mathcal{A}_\infty = \mathcal{A} \cup \{H_\infty\}$



H_∞

$$\lambda_\infty := -\sum_{i=1}^{n} \lambda_i$$

$(Y_{\mathcal{A}})$ If $\lambda_X \neq 0 \ \forall X \in L(\mathcal{A})$ then

$$H^k(A, a_\lambda) = \begin{cases} 0 & k \neq \ell \\ \mathbb{C}^{\beta(\mathcal{A})} & k = \ell \end{cases}$$

$\beta(\mathcal{A})$ = beta invariant of $c\mathcal{A}$

$$= \left| \frac{\mathrm{Poin}(L(c\mathcal{A}), t)}{(1+t)} \Big|_{t=-1} \right|$$

$= |\chi(M(\mathcal{A}))| \underset{\text{Zaslavsky}}{\overset{\text{Thm}}{=}}$ # Bdd Chambers of $M(\mathcal{A}) \cap \mathbb{R}^\ell$ if $\mathcal{A}$ is complexified real

II. "Gauge Transformations"

① $\lambda, \lambda' \in \mathbb{C}^n$ and $\lambda - \lambda' \in \mathbb{Z}^n$

$$\Rightarrow \mathcal{L}_t = \mathcal{L}_{t'} \Rightarrow H^*(M, \mathcal{L}_t) = H^*(M, \mathcal{L}_{t'})$$

But $H^*(A, a_\lambda)$ & $H^*(A, a_{\lambda'})$ are not necessarily ~~equal~~ iso.

② If $\lambda' = c\lambda$, $c \in \mathbb{C}^*$ then $H^*(A, a_\lambda) \cong H^*(A, a_{\lambda'})$

but $H^*(M, \mathcal{L}_t)$ & $H^*(M, \mathcal{L}_{t'})$ are not necessarily ~~equal~~ iso.

Note: (1) each H_i is irreducible

(2) $rk(X)=2 \Rightarrow$, X is irreducible iff $|\mathcal{A}_X| \geq 3$.

Modifications of λ:

Assume $\lambda_i \notin \mathbb{Z}$ $\forall i$ and $\sum_{i=1}^{n} \lambda_i \notin \mathbb{Z}$

Replace λ_i with $\underbrace{\lambda_i - N}_{\lambda_i'}$, $N \gg 0$, $N \in \mathbb{Z}$ for $i=1,\dots,n$

Then $\lambda_X' \notin \mathbb{Z}_{>0}$ for any $X \not\subseteq H_\infty$

$$\lambda_\infty' = \left(-\sum_{i=1}^{n} \lambda_i\right) + nN$$

If no $X \subseteq H_\infty$, $X \neq H_\infty$, is irr., then ESV/STV cond. is satisfied.

$$H^*(M, \mathcal{L}_t) \cong H^*(A, a_{\lambda'})$$

Degree One Resonance Varieties

Assume $\ell \geq 2$

For what λ is $H^1(A, a_\lambda) \neq 0$?

$\mathcal{R}^1(\mathcal{A}) = \{\lambda \in \mathbb{C}^n \mid H^1(A, a_\lambda) \neq 0\}$ $\quad \mathbb{K}$ any field

$\mathcal{R}^1_d(\mathcal{A}) = \{\lambda \in \mathbb{C}^n \mid \dim_{\mathbb{K}} H^1(A, a_\lambda) \geq d\}$

$$0 \to A_0 \xrightarrow{a_\lambda} A_1 \xrightarrow{a_\lambda} A_2 \xrightarrow{a_\lambda} \cdots$$
$$A_0 = \mathbb{K}$$

$a_\mu \in A_1$ represents a non-trivial element of $H^1(A, a_\lambda)$ iff $a_\lambda \wedge a_\mu = 0$ and $a_\mu \notin \mathbb{K} a_\lambda$ (i.e. $\mu \notin \mathbb{K}\lambda$)

Lifting to E:

$$0 \neq e_\lambda \wedge e_\mu \in I^2 \Leftrightarrow 0 \neq [a_\mu] \in H^1(A, a_\lambda)$$

Recall: Assume $\mathcal{A}$ is central (of rank 3)

$$A_2 = \bigoplus_{\substack{X \in L \\ rk(X)=2}} A_X^2$$

Write $a_\lambda^X = \sum_{H_i \geq X} \lambda_i a_i$

Then $a_\lambda \wedge a_\mu$ iff $a_\lambda^X \wedge a_\mu^X = 0$ $\forall X \in L,\ rk(X) = 2$

Problem 1: Sps $\mathcal{A}$ has rk two. Show

$a_\lambda \wedge a_\mu = 0$ iff $\mu \in \mathbb{K}\lambda$ or $\underbrace{\sum_{i=1}^{n} \lambda_i = 0 = \sum_{i=1}^{n} \mu_i}_{n \geq 3 \text{ and}}$

(Hint: Recall $\partial e_{ijk} = (e_i - e_j) \wedge (e_j - e_k)$)

Thm: $\lambda \in R^1(A, \mathbb{K})$ iff $\exists \mu \in \mathbb{K}^n$ s.t. $\forall X \in L(\mathcal{A}),\ rk(X) = 2$ either (i) $\mu^X \parallel \lambda^X$ and $\lambda \nparallel \mu$

or (ii) $|\mathcal{A}_X| \geq 3$ and $\lambda_X = \mu_X = 0$.

$$\lambda^X = (\lambda_i \mid H_i \in \mathcal{A}_X) \in \mathbb{K}^{|\mathcal{A}_X|}$$

Assume λ & μ satisfy theorem then we say (λ, μ) is a <u>resonant pair</u>.

Define $\operatorname{supp}(\lambda,\mu) = \{ i \in [n] \mid \lambda_i \neq 0 \text{ or } \mu_i \neq 0 \}$

Assume $\operatorname{supp}(\lambda,\mu) = [n]$

Define an equivalence relation on $[n]$ by $i \sim j$ iff $\begin{vmatrix} \lambda_i & \mu_i \\ \lambda_j & \mu_j \end{vmatrix} = 0$.

The Theorem now says:

$\forall X \in L$, $\operatorname{rk}(X) = 2$ either

(i) $i \sim j \ \forall i, j \in X$

or (ii) $|X| \geq 3$ and $\lambda_X = 0 = \mu_X$

<u>Lemma</u>: Sps $\mathcal{A}$ has rk two. Then A_2 has ~~bas~~ basis

$$\{ a_1 \wedge a_j \mid 2 \leq j \leq n \}$$

<u>Pf</u>: Exercise

<u>Thm</u>: Sps $X = \{1, \dots, p\}$ and $\mathrm{rk}\, X = 2$

$= \{1, \dots, p\}$

$i \sim j \quad \forall \; i, j \geq 2$

Then $\quad i \sim j \quad \forall \; i, j \geq 1$

<u>Pf</u>:
$$0 = \underbrace{\Big(\sum_{i=1}^{p} \lambda_i a_i\Big)}_{a_\lambda^X} \wedge \underbrace{\Big(\sum_{j=1}^{p} \mu_j a_j\Big)}_{a_\mu^X} = \sum \begin{vmatrix} \lambda_i & \mu_i \\ \lambda_j & \mu_j \end{vmatrix} a_i \wedge a_j = 0$$

$$= \sum_j \begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_j & \mu_j \end{vmatrix} a_1 \wedge a_j$$

By lemma $\begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_1 & \mu_1 \end{vmatrix} = 0 \quad \forall j$.

<u>Def</u>: A partition P of $[n]$ is <u>neighborly</u> (w.r.t. $\mathcal{A}$) iff for every block B of P and " " flat X of $\mathrm{rk}\, X = 2$

$|X \cap B| \geq |X| - 1$ then $X \subseteq B$

<u>Cor.</u> If (λ, μ) is a resonant pair, then the induced partition of $[n]$ is neighborly.

<u>Cor.</u> If (λ, μ) is a resonant pair and X is a flat of $\mathrm{rk}\, X = 2$ with $X \not\subseteq$ any block of the induced partition then $\lambda_X = 0 = \mu_X$.

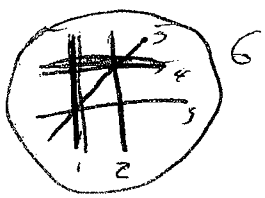
So, we have:

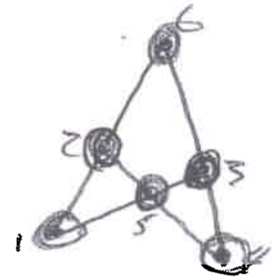
If ~~X∈B'~~ $\mathcal{A}$ supports a resonant pair of weights (λ,μ) then $\mathcal{A}$ supports a neighborly partition P and

$\exists$ two independent vectors in the kernel of the incidence matrix

$$\begin{array}{r c} & 1 \; - \; - \; - \; - \; n \\ X \not\subseteq B \rightarrow & \left[\begin{array}{ccc} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{array} \right] \\ \text{"multi-colored} \rightarrow & \\ \text{flats"} \rightarrow & \end{array}.$$

Ex:





$P = 14 \mid 23 \mid 56$

★ Find examples of arr.'s that support neighborly partitions.

Coroll.

If $|X \cap B| \geq |X| - 1 \Rightarrow X \subseteq B$

If $|X| = 2$ then $X \cap B \neq \emptyset \Rightarrow X \subseteq B$.

$$\begin{array}{r c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{r} 126 \\ 135 \\ 245 \\ 346 \end{array} & \left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right] \end{array}$$

has nulli ≥ 2

$\Rightarrow \quad \lambda = (1,0,0,1,-1,-1) = \text{red} - \text{green}$

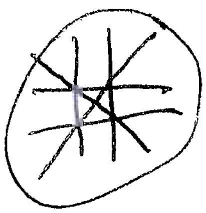
$\mu = (0,1,1,0,-1,-1) = \text{blue} - \text{green}$

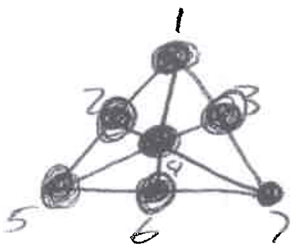
Conclusion: $(a_1+a_4-a_5-a_6)\wedge(a_2+a_3-a_5-a_6)=0$

$a_\lambda \wedge a_\mu = 0$

$\Rightarrow \mathbb{K}a_\lambda + \mathbb{K}a_\mu \leq R^1(A)$

Ex:





$P = 1|236|4|5|7$

6×7

$$\underset{6}{I} = \begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 125 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 137 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 247 & & & & & & & \\ \vdots & & & & & & & \end{array}$$

$\text{nullity}_{\mathbb{C}}(I) = 1$

$\text{nullity}_{\mathbb{Z}_2}(I) = 3$

nullspace has basis: ________

over $\mathbb{Z}_2$, there is a vector in $R^1_2(A)$